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REVIEW ARTICLE

Weimin DAI, Jian-Qiang HU, Lei LEI

Discrete-event stochastic systems with correlated inputs: Modeling and performance evaluation

Abstract In the majority of the previous works on discrete-event stochastic systems, they have been assumed to have independent input processes. However, in many applications, these input processes can be highly correlated. Furthermore, the performance measures of the systems with correlated inputs can be significantly different from those with independent inputs. In this paper, we provide an overview on some commonly used methods for modeling correlated input processes, and we discuss the difficulties and possible future research topics in the study of discrete-event stochastic systems with correlated inputs.

Keywords discrete-event stochastic system, correlated input, performance evaluation

1 Introduction

Traditionally, in the study of discrete-event stochastic systems, the input processes that drive these systems are assumed to be renewal processes, that is, they have independent and identically distributed (i.i.d.) inter-event times. However, in many applications, the input processes of discrete-event stochastic systems are often correlated (e.g., inter-arrival times and service times of customers in queuing networks (Szekli et al., 1994; Choi et al., 2008) and the demands over multiple periods in inventory systems (Shang, 2012; Hu et al., 2016)). Diaz et al. (2016) indicated that advertising campaigns may create a dependence effect (e.g., induced autocorrelation) on the probabilistic demand. Carrizosa et al. (2016) indicated that demand is usually correlated along time; thus, assuming that demands over multiple periods are independent is practically unrealistic. Capturing the dependence of these correlated inputs is important because it has significant influence on performance measures. Furthermore, in some cases, one might be interested in the effect of the correlation on performance measures. Unfortunately, studies on discrete-event stochastic systems with correlated inputs are few. In dealing with such systems, we face two major difficulties: One is how to model correlated inputs, and the other is that readily available analytical methods are limited.

In this review paper, we want to give a brief introduction on some of the commonly used models to characterize correlated input processes (e.g., autoregressive (AR), AR-moving-average (ARMA), and AR-to-anything (ARTA) processes; transform expand sample (TES) models; Markov-modulated processes; and copula-based models) and compare their advantages and disadvantages. More importantly, we want to discuss some of the recent works and explore possible future research directions in this area, so that more researchers may become interested in pursuing related research topics.

First, we are interested in how to model a time series X_n, which can be thought as an input process for a discrete-event stochastic system (e.g., inter-arrival times or service times in a queue or demands over multiple periods of an inventory system). Particularly, we assume that X_n is correlated. In stochastic simulation, and particularly in the framework of input modeling, generating a time series with a given correlation structure (e.g., lagged autocorrelations) and marginal distribution is a related problem considered in many studies (Kuhl et al., 2010; Kugiumtzis and Bora-Senta, 2014; Bardsley, 2017). Thus, we would like to model X_n, for example, under the assumption that its marginal distributions and lag-1 autocorrelation are given. We would also like to know how to analyze a stochastic system with X_n (or maybe several series of X_n) as its inputs, in which case

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Weimin DAI, Jian-Qiang HU
School of Management, Fudan University, Shanghai 200433, China

Lei LEI (✉)
School of Economics and Business Administration, Chongqing University, Chongqing 400044, China
E-mail: leilei312@cqu.edu.cn

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we would use a G/G/1 queue (an infinite-capacity queuing system with general independent input, general service process and a single server) as an illustrative example. For the G/G/1 queue, the inter-arrival time between customers n and $n+1$, denoted as A_n, and the service time of customer n, denoted as S_n, can be regarded as two streams of input processes, and they can also be correlated with each other. In this setting, we are interested in some performance measures of the G/G/1 queue, such as the average waiting time of a customer.

The rest of the paper is organized as follows. Section 2 gives an overview on four methods for modeling correlated processes and discusses their advantages and disadvantages. Section 3 discusses how to model and analyze discrete-event stochastic systems with correlated inputs. Section 4 presents some possible research directions in this area.

2 Models for correlated inputs

In this section, we present four models for correlated input processes. They are AR/ARMA/ARTA processes, TES processes, Markov-modulated processes, and copula-based processes. We also discuss some of their advantages, disadvantages, and relationships.

2.1 AR, ARMA, and ARTA processes

AR process, which assumes the normality of an input distribution, is perhaps the most widely used one in the literature for modeling time series data with correlation. An AR process of order p, denoted as AR(p), is defined as (Box et al., 1970)

$$X_n = \alpha_1 X_{n-1} + \alpha_2 X_{n-2} + \ldots + \alpha_p X_{n-p} + \epsilon_n, \tag{1}$$

where ϵ_n is normally distributed with a mean of 0 and variance of σ_ϵ^2. In the AR(p) model, X_n is expressed as a weighted sum of X_i $(i = n-p, \ldots, n-1)$ plus an innovation ϵ_n. The AR(p) model can be extended to an ARMA(p, q) model, which combines the AR model with a moving average of previous q innovations, that is,

$$X_n = \alpha_1 X_{n-1} + \ldots + \alpha_p X_{n-p} + \epsilon_n - \theta_1 \epsilon_{n-1} - \ldots - \theta_q \epsilon_{n-q}. \tag{2}$$

AR and ARMA processes can generate random samples with a normal marginal distribution but cannot be used in problems in which input variables are not normal. ARTA processes are therefore introduced to model other types of marginal distribution (Cario and Nelson, 1998). Briefly, to construct an ARTA process $\{Y_n\}$, we initially apply the standard normal cumulative distribution function Φ on X_n and then use the inverse transformation method to obtain $Y_n = F_Y^{-1}(\Phi(X_n))$, with X_n being the AR(p) process defined by Eq. (1) and F_Y being the marginal distribution of Y_n. The variance of ϵ_n in the ARTA process is set to be

$$\sigma_\epsilon^2 = 1 - \alpha_1 \mathrm{Corr}(X_n, X_{n+1}) - \alpha_2 \mathrm{Corr}(X_n, X_{n+2}) - \ldots - \alpha_p \mathrm{Corr}(X_n, X_{n+p}), \tag{3}$$

so that the marginal distribution of X_n is the standard normal; hence, $\Phi(X_n)$ is uniformly distributed over [0, 1). With this, we can obtain an ARTA process for any given marginal distribution F_Y. For example, for the G/G/1 queue, the inter-arrival times can be constructed as $F_a^{-1}(\Phi(X_n))$, or the service times as $F_s^{-1}(\Phi(X_n))$.

Due to their simplicity and linearity, AR and ARMA processes have been widely used in the literature. In some cases, they are used for analytical analysis. However, the main drawback is their requirement of normal marginal distribution. ARTA processes are developed to overcome this difficulty, but the dependence structure they can capture is still quite limited. In fact, an ARTA process for fitting a specified lag-1 autocorrelation is considerably difficult to construct. To summarize, the three processes are mainly suitable for modeling linear correlation but are not highly appropriate for other dependence structures, such as tail dependence.

2.2 TES models

We now present the second model for correlated processes: TES. TES was first introduced by Melamed (1991) and investigated in detail later by Jagerman and Melamed (1992). The basic idea of TES is to generate two correlated processes $\{U_n^+\}$ and $\{U_n^-\}$, which have a uniform marginal, and then transform them to variables with arbitrary marginal distributions via the inversion method. Therefore, generating $\{U_n^+\}$ and $\{U_n^-\}$ is the key in TES, and they are also called TES background sequences.

We now illustrate how to construct $\{U_n^+\}$ and $\{U_n^-\}$. A basic TES model is parameterized by a pair of parameters (L, R): $-0.5 < L < R \leqslant 0.5$. The function of these parameters is to achieve the full range coverage of lag-1 autocorrelation: $\{U_n^+\}$ covers the positive range [0, 1] and $\{U_n^-\}$ covers the negative range [−1, 0]. This function also motivates the use of superscripts "+" and "–" in TES modeling.

For any real number x, let $\lfloor x \rfloor \doteq \max\{\text{integer } n : n \leqslant x\}$ denote the integral part of x, and $\langle x \rangle \doteq x - \lfloor x \rfloor$ denote the fractional part of x. Let $\{V_n\}$ be a sequence of i.i.d. uniform random variables on [0, 1). Furthermore, let U_0 be a uniform random variable on [0, 1) and assume that it is independent of $\{V_n\}$. Sequences $\{U_n^+\}$ and $\{U_n^-\}$ are recursively defined by

$$U_n^+ = \begin{cases} U_0, & n = 0 \\ \langle U_{n-1}^+ L + (R-L) V_n \rangle, & n > 0 \end{cases},$$
$$U_n^- = \begin{cases} U_n^+, & \text{if } n \text{ is even} \\ 1 - U_n^+, & \text{if } n \text{ is odd} \end{cases}. \tag{4}$$

The TES background processes $\{U_n^+\}$ and $\{U_n^-\}$ are stationary Markovian processes with uniform marginals on [0, 1), and $\{V_n\}$ only influences the second-order properties of the TES process (Melamed and Hill, 1995). The foreground sequences of TES processes, $\{X_n^+\}$ and $\{X_n^-\}$, can be obtained by the inverse transformation. For instance, if $F(x)$ is the marginal distribution of the input process, then processes $\{X_n^+\}$ and $\{X_n^-\}$ can be obtained by letting $X_n^+ = F^{-1}(U_n^+)$ and $X_n^- = F^{-1}(U_n^-)$, respectively.

Autocorrelation functions $\rho_U^+(\tau)$ and $\rho_U^-(\tau)$ are defined as

$$\rho_U^+(\tau) = \frac{\mathrm{E}[X_n^+ X_{n+\tau}^+] - \mu_X^2}{\sigma_X^2},\ \rho_U^-(\tau) = \frac{\mathrm{E}[X_n^- X_{n+\tau}^-] - \mu_X^2}{\sigma_X^2},\quad \tau = 1, 2, \ldots, \tag{5}$$

where μ_X and σ_X are the mean and variance of $\{X_n^+\}$ and $\{X_n^-\}$, respectively (notice that $\{X_n^+\}$ and $\{X_n^-\}$ have the same mean and variance). One advantage of the TES model is that it can accommodate autocorrelation functions with various shapes. Particularly, when $R + L = 0$, the resultant autocorrelation function is monotonically decreasing to zero with respect to τ, and when $R + L \neq 0$, the autocorrelation function is oscillatory with respect to τ with envelopes converging to zero.

TES can be applied to model a broad class of correlated time series inputs with general marginal distributions and various dependence structures. It has the ability to simultaneously fit any given marginal distribution and lag-1 autocorrelation. It can also cover a wide range of autocorrelation functions, including monotone and oscillatory ones. Computationally, TES sequences are fairly easy and efficient to generate; hence, TES is highly suitable for Monte Carlo simulation (Melamed, 1993). However, similar to AR/ARMA/ARTA processes, TES is mostly effective when autocorrelations are linear, and using it to model correlated inputs with tail dependence is difficult.

2.3 Markov-modulated processes

Another popular method used to model correlated processes is based on Markov-modulated processes. A Markov-modulated process is a stochastic process whose correlation is induced by the underlying Markov process, and it can be defined as follows.

Let $\{M_n\}$ be an irreducible discrete-time Markov process with state space S and transition probability matrix $\boldsymbol{P} = (p_{ij})_{i,j\in S}$. Let $F_{ij}(x)$ be the distribution function of X_n, given that $M_{n-1} = i$, $M_n = j$, that is,

$$F_{ij}(x) = \Pr(X_n \leqslant x \mid M_{n-1} = i,\ M_n = j). \tag{6}$$

We then say X_n is a Markov-modulated process driven by $\{M_n\}$.

Markov-modulated processes were first introduced to model correlated inter-arrival and service times for queues (Neuts, 1979). A special case of Markov-modulated processes is the Markov-modulated Poisson process (MMPP), in which $F_{ij}(x)$ is an exponential distribution for all $i, j \in S$. MMPP can also be thought as a doubly stochastic Poisson process with rate depending on a Markov process $\{M_n\}$. Markov-modulated processes can be further extended to batch Markov-modulated processes to allow multiple events occurring at once (Lucantoni, 1991). For a thorough review of Markov-modulated processes, the reader is referred to Cesar (2015).

Girish and Hu (1999) discussed how to fit a Markov-modulated process to the departure process of the G/G/1 queue, $\{D_n\}$. Suppose $\{D_n\}$ has the k-th marginal moment of m_k and the lag-1 autocorrelation r (which are given). If we want to use a Markov-modulated process X_n to fit $\{D_n\}$, then we have

$$m_k = \sum_{i\in S}\sum_{j\in S}\pi_i p_{ij} a_{ijk},\ k = 1, 2, \ldots, K,$$

$$r = \sum_{i\in S}\sum_{j\in S}\sum_{\ell\in S}\pi_i p_{ij} p_{j\ell} a_{ij1} a_{j\ell 1} - m_1^2, \tag{7}$$

where π_i is the steady state of the underlying Markov process $\{M_n\}$, and a_{ijk} is the conditional moments of X_n defined by $a_{ijk} = \mathrm{E}[X_n^k \mid M_{n-1} = i,\ M_n = j]$. We have $((K+1)|S|^2 - |S|)$ parameters with $K+1$ equations. Therefore, we have considerable flexibility (or uncertainty) to construct the Markov-modulated process. Given that Eq. (7) is a set of nonlinear equations, it is generally difficult to handle.

Markov-modulated processes preserve many traditional Markovian properties; thus, analysis under the framework of a Markov process is relatively easy. However, the Markov-modulated processes have two main shortcomings. First, the effects of correlation are difficult to be separated from those of marginal distributions. Second, Markov-modulated processes are often overparameterized for statistical purposes.

2.4 Copula-based models

A copula function is a multivariate probability distribution function with uniform margins. Formally, a d-dimensional copula $C: [0, 1]^d \to [0, 1]$ is the cumulative probability function of a random vector $(U_1, \ldots, U_d)$ with a uniform margin, as shown as follows:

$$C(u_1, \ldots, u_d) = \Pr(U_1 \leqslant u_1, \ldots, U_d \leqslant u_d). \tag{8}$$

If X_i is a random variable with continuous probability function $F_i(x)$, then $U_i = F_i(X_i)$ is uniformly distributed on [1, 0). This condition enables us to relate $(F_1(X_1), \ldots, F_d(X_d))$ to a copula function. Particularly, the following results are very useful (Nelsen, 2006).

1) Let C be a d-dimensional copula function, and

$F_1, \ldots, F_d$ be univariate cumulative distribution functions. Then, the function

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \tag{9}$$

for all $(x_1, \ldots, x_d) \in \bar{R}^d (d \geqslant 2)$ is a d-dimensional cumulative distribution function with margins $F_1, \ldots, F_d$.

2) Let H be a d-dimensional cumulative distribution function, and $F_1, \ldots, F_d$ be continuous univariate cumulative distribution functions. Then, there exists a unique d-dimensional copula, such that

$$C(u_1, \ldots, u_d) = H(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \tag{10}$$

for all $(u_1, \ldots, u_d) \in [0, 1]^d (d \geqslant 2)$.

The above results are usually referred as Sklar's theorem, based on which we have: For any marginal distribution F_i $(i = 1, \ldots, d)$ and a copula C, there exists a joint distribution function H, such that Eq. (9) holds. On the other hand, for any given joint probability distribution function H, there exists a unique copula, such that Eq. (10) holds if the marginals are continuous. The concept of copulas allows us to decouple the dependence structure among random variables from their marginal distributions. This function gives us the flexibility of constructing random variables with given marginal probability distributions in conjunction with various types of dependence structure, and it makes studying the effects of correlations considerably easier.

Although most of the previous works on copulas have focused on modeling the contemporaneous dependence among multiple random variables, a stream of literature has also used copulas to model the temporal dependence of univariate time series inputs. Darsow et al. (1992) provided the characterizations of first-order[1)] Markov processes in terms of two-dimensional copula, and Ibragimov (2009) obtained the characterizations of a copula-based time series to be a higher-order Markov process.

We now illustrate how to construct a Markov process based on copulas. Let $\{Y_t\}$ be a Markov process with a continuous marginal distribution G. The process can be fully characterized by the bivariate joint distribution of $\{Y_{t-1}\}$ and $\{Y_t\}$, say, $F(y_1, y_2)$. On the basis of Sklar's theorem, the joint distribution function $F(\cdot, \cdot)$ can be uniquely expressed in terms of the marginal distribution G and a bivariate copula function $C(\cdot, \cdot)$, as shown as follows

$$F(y_1, y_2) = C(G(y_1), G(y_2)). \tag{11}$$

Thus, we can completely specify a Markov process with its marginal distribution and a bivariate copula.

In most cases, simulation is the only tool available to study copula-based processes/systems. Here the first key question is how to generate random variables with copula distribution functions. It is nontrivial and often computationally intensive. Suppose $(U_1, \ldots, U_d)$ have joint copula distribution function C. Then, the conditional distribution of U_k, given the values of $(U_1, \ldots, U_{k-1})$ is

$$\begin{aligned} &C_k(u_k \mid u_1, \ldots, u_{k-1}) \\ &= \Pr(U_k \leqslant u_k \mid U_1 \leqslant u_1, \ldots, U_{k-1} \leqslant u_{k-1}) \\ &= \frac{\partial^{k-1} C(u_1, \ldots, u_k, 1, \ldots, 1)}{\partial u_1 \ldots \partial u_{k-1}} \bigg/ \frac{\partial^{k-1} C(u_1, \ldots, u_{k-1}, 1, \ldots, 1)}{\partial u_1 \ldots \partial u_{k-1}}. \end{aligned} \tag{12}$$

Hence, $(U_1, \ldots, U_d)$ can be generated as follows.

- Generate a random variate U_1 from $U(0, 1)$.
- Generate a random variate U_2 from $C_2(\cdot \mid U_1)$.
- …
- Generate a random variate U_d from $C_d(\cdot \mid U_1, \ldots, U_{d-1})$.

With $(U_1, \ldots, U_d)$, any random variables $(X_1, \ldots, X_d)$ can then be generated as

$$(F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d)), \tag{13}$$

as per Sklar's theorem, where F_i is the marginal distribution of X_i $(i = 1, \ldots, d)$. However, generating random variables in such a way often requires considerable computation efforts. First, the conditional distribution $C_k(u_k \mid u_1, \ldots, u_{k-1})$ does not generally have an analytical formula, and a root-finding algorithm may have to be used. To overcome this difficulty, Marshall and Olkin (1988) proposed an alternative approach to more efficiently generate samples for Archimedean copulas, an important family of copulas. Second, the inversion calculation of marginal distribution functions can be difficult in many cases, such as for Erlang and hyperexponential distributions.

The copula-based method has two clear advantages in modeling correlated input processes. First, it can completely separate marginal distributions and correlations via copula functions. Second, the availability of many different copula families makes it possible to model different types of dependence, including nonlinear, asymmetric, and tail dependence. In comparison, the ARMA process can only capture linear dependence in correlation. Lei et al. (2022) showed that even at the same level of linear correlation in input processes, system performances may be quite different, and copula-based models are more advantageous in capturing and fitting different structures of dependence.

2.5 Relationship among the four models

Some of the four models are in fact related to one another. For example, an ARTA process can be viewed

1) A process $\{X_t\}_{t\in T}$ is called as Markov process of order $k \geqslant 1$ if, for all $t_1 < \ldots < t_{n-k+1} < \ldots < t_n < t$, $\Pr(X_t < x \mid X_{t_1}, \ldots, X_{t_{n-k+1}}, \ldots, X_{t_n}) = \Pr(X_t < x \mid X_{t_{n-k+1}}, \ldots, X_{t_n})$.

as a special case of the copula-based model. Similar to ARTA, Cario and Nelson (1998) also developed the method, Normal-To-Anything (NORTA), for random vectors. On the basis of this work, Biller and Nelson (2003; 2008) developed the method of vector ARTA (VARTA). The Gaussian copula is actually the key ingredient in these transformation-based models. Biller (2009) illustrated the shortcomings of VARTA and extended it to general copulas instead of the specific Gaussian copula. The copula-based model and Markov processes are also closely related (Lei et al., 2022). Lei et al. (2022)'s study showed that discrete copula-based processes can be transformed to Markov-modulated processes. Conversely, TES and copula-based models use the similar idea in their construction of correlated processes, that is, they initially generate correlated process with uniform marginal distribution and then use the inversion method in their transformation.

3 Systems with correlated inputs

Lei et al. (2022) proposed to use the framework of generalized semi-Markov process (GSMP) to model discrete-event stochastic systems with correlated inputs. Although it focuses on copula-based correlated input processes, it can be applied to any of the four models discussed in this paper. For details of the GSMP framework, the reader is referred to Lei et al. (2022). Essentially, GSMP provides a formal way to simulate or generate sample paths of discrete-event stochastic systems. It is particularly useful when the method of simulation is needed. Generally, obtaining analytical results for discrete-event stochastic systems with correlated inputs is considerably difficult. Therefore, in most cases, simulation is probably the only tool available.

AR/ARMA/ARTA models have been used to study the effect of autocorrelation. For example, they are used to model correlated demands in investigating information sharing and demand propagation, where sharing sales information has been viewed as a major strategy to counter the so-called "bullwhip effect" (Lee et al., 1997). Lee et al. (2000) found that the value of sales information sharing can be high especially when demands are significantly correlated over time, modeled by an $AR(1)$ process, because when correlation is large, current demand information is more valuable for predicting future demands. Zhang (2004) and Gaur et al. (2005) extended the work of Lee et al. (2000) by studying the value of information sharing in a supply chain where the retailer serves an $AR(p, q)$ demand as opposed to an $AR(1)$ demand. Information sharing and demand propagation have been further investigated by using ARMA to model the demands (Giloni et al., 2014). Pereira et al. (2012) provided an analysis of the effect of autocorrelation controlled by parameters in AR processes on the performance of a manufacturing process based on simulation.

Livny et al. (1993) used simulation to investigate the effect of correlation in inter-arrival and service times of queuing systems based on the TES model. They showed that the correlation in inter-arrival and service times has a significant effect on the average waiting time. A similar study on manufacturing systems was performed by Altiok and Melamed (2001). For the G/G/1 queue with Markov-modulated arrival time, Szekli et al. (1994) showed that the average waiting time increases with respect to the autocorrelation in the inter-arrival times.

Markov-modulated processes have been widely used in queuing networks for correlated inter-arrival and service times and inventory systems for multi-period demands. Runnenburg (1962) investigated the effect of correlation on the average waiting time for the G/G/1 queue with an integer-valued Markov-modulated arrival process and exponential service times. Fischer and Meier-Hellstern (1993) demonstrated that the MMPP is quite useful for modeling correlated arrival processes because it captures the important correlations while still remaining analytically tractable. Patuwo et al. (1993) used the Markov-modulated arrival process to study the effects of serial correlations in the arrival process. Patuwo et al. (1993) studied the G/G/1 queue with a two-state Markov-modulated inter-arrival process with the mixture of Erlang marginal probability distribution. Fischer and Meier-Hellstern (1993) derived a number of analytical results, including queue length distribution and waiting time distribution for the G/G/1 queue with Markov-modulated Poisson arrival process. Szekli et al. (1994) showed that the average waiting time increases with respect to the autocorrelation in the inter-arrival times for the G/G/1 queue with Markov-modulated arrival time. Lim et al. (2006) studied the departure process of the G/G/1 queue with Markov-modulated Poisson arrival process. For the G/G/1 queue with Markov-modulated inter-arrival and service times, the method of Wiener–Hopf factorization was developed to analyze the average waiting time; however, it is not highly effective because it relies on solving a system of integral equations. Zhu and Li (1993) developed an effective algorithm to calculate the moments of the waiting time based on the method of MacLaurin series analysis, a method first proposed by Gong and Hu (1992) to analyze a traditional G/G/1 queue with i.i.d. inter-arrival and service times. Hu (1996) extended the method to analyze the departure process of the queue. Hu et al. (2016) applied the same method to an (s, S) inventory system with Markov-modulated demands to calculate the moments of the inventory level, based on which various performance measures of the system can also be evaluated.

As mentioned earlier, the copula-based method can be used to model various dependence, including nonlinear, asymmetric, and tail dependence; hence, it has attracted considerable attention in recent years. It has been widely used in financial and insurance risk managements (Frey and McNeil, 2003; Embrechts, 2009). The wide range of copula families and their ease to fit arbitrary marginal distributions enable one to select the desirable copulas that satisfy the required properties or fit empirical data. The benefits of using copulas in modeling dependence structures of inputs and the review of techniques to construct copula-based input models representing positive tail dependencies are referred to Biller (2009) and Biller and Gunes Corlu (2012). Biller (2009) proposed a copula-based multivariate time-series input model and developed efficient fitting and sampling algorithms for the model, which are suited for driving large-scale stochastic simulation. Jaoua et al. (2013) used a copula-based approach to model a type of asymmetric dependence structure, which is found in empirical data, and explored the sensitivity of the pooling decision in a multi-skill call-center with respect to the dependence. Their simulation results showed that the assumption of independence, as well as the misspecification of the dependence structure, can lead to substantial errors in call-center performance.

Some attempts have been made recently to develop analytical tools for discrete-event stochastic systems with correlated inputs; the use of MacLaurin series analysis to study queuing and inventory systems with Markov-modulated inputs is one example. For systems with copula-based correlated inputs, Lei et al. (2022) proposed a method based on discretization, which converts such systems into systems with Markov-modulated inputs. Nevertheless, considerable works remain to be done.

4 Future research directions

We believe that several exciting future research directions are worth pursuing in this area, as listed as follows.

- Analytical tools should be developed in the study of discrete-event stochastic systems with correlated inputs. As mentioned earlier, analytical results for discrete-event stochastic systems with correlated inputs are generally extremely difficult to obtain. Simulation is probably the only method available in most cases. However, two methods seem to be promising. One is to use the theory of Markov process. For example, if we use Markov-modulated processes as inputs, then we can most likely model the resulting discrete-event stochastic systems as Markov systems. One major difficulty is perhaps the large size of state spaces, which can make analytical results infeasible to obtain. Another method is based on the technique of MacLaurin series, as discussed in Section 3. Previous works that have mainly focused on Markov-modulated correlated processes would be interesting to extend to other types of correlated process.
- For most problems in the area of discrete-event stochastic systems, input processes are traditionally assumed to be independent. Therefore, what if independent inputs are replaced by correlated inputs is an idea worth exploring.
- The effects of the correlations of input processes on discrete-event stochastic systems should be studied. Here, the key is to select suitable correlated input models to obtain analytical results. Simulation can always be used as the last resort in the study of this type of problem.
- Efficient computation methods for constructing copula-based correlated inputs should be developed. We believe the copula-based model is the best among the existing correlated models for correlated input processes in terms of its flexibility to separate correlations from marginal distributions and its ability to cover a wide range of correlation structures. However, handling copula-based correlated processes, particularly when simulation is the only tool available, is relatively more difficult because generating samples for copula random variables often requires significant amount of computation efforts. Therefore, efficient methods must be developed to generate a sample of random variables with copula distributions.
- Good methods must be developed to fit a correlated input model to real data. Although we have several different models for correlated processes, we lack efficient statistical methods to fit these correlated models to real data. Unless this issue can be resolved, applying these models in real-world problems will be difficult.
- The applicability of correlated input models must be studied based on real data in various applications. Currently, which correlated model is more suitable for correlated input processes in many various application problems remain unclear. Therefore, more empirical studies are needed.

References

Altiok T, Melamed B (2001). The case for modeling correlation in manufacturing systems. IIE Transactions, 33(9): 779–791

Bardsley E (2017). A finite mixture approach to univariate data simulation with moment matching. Environmental Modelling & Software, 90: 27–33

Biller B (2009). Copula-based multivariate input models for stochastic simulation. Operations Research, 57(4): 878–892

Biller B, Gunes Corlu C (2012). Copula-based multivariate input modeling. Surveys in Operations Research and Management Science, 17(2): 69–84

Biller B, Nelson B L (2003). Modeling and generating multivariate time-series input processes using a vector autoregressive technique. ACM Transactions on Modeling and Computer Simulation, 13(3): 211–237

Biller B, Nelson B L (2008). Evaluation of the ARTAFIT method for fitting time-series input processes for simulation. INFORMS Journal on Computing, 20(3): 485–498

Box G E, Jenkins G M, Reinsel G C, Ljung G M (1970). Time Series Analysis: Forecasting and Control. Hoboken, NJ: John Wiley & Sons

Cario M C, Nelson B L (1998). Numerical methods for fitting and simulating autoregressive-to-anything processes. INFORMS Journal on Computing, 10(1): 72–81

Carrizosa E, Olivares-Nadal A V, Ramírez-Cobo P (2016). Robust newsvendor problem with autoregressive demand. Computers & Operations Research, 68: 123–133

Cesar J V R (2015). Markovian Arrival Processes: The Identifiability Issue and Other Applied Aspects. Dissertation for the Doctoral Degree. Leganés: Universidad Carlos III de Madrid

Choi D I, Kim T S, Lee S (2008). Analysis of an MMPP/G/1/K queue with queue length dependent arrival rates, and its application to preventive congestion control in telecommunication networks. European Journal of Operational Research, 187(2): 652–659

Darsow W F, Nguyen B, Olsen E T (1992). Copulas and Markov processes. Illinois Journal of Mathematics, 36(4): 600–642

Diaz R, Bailey M P, Kumar S (2016). Analyzing a lost-sale stochastic inventory model with Markov-modulated demands: A simulation-based optimization study. Journal of Manufacturing Systems, 38: 1–12

Embrechts P (2009). Copulas: A personal view. Journal of Risk and Insurance, 76(3): 639–650

Fischer W, Meier-Hellstern K (1993). The Markov-modulated Poisson process (MMPP) cookbook. Performance Evaluation, 18(2): 149–171

Frey R, McNeil A (2003). Dependent defaults in models of portfolio credit risk. Journal of Risk, 6(1): 59–92

Gaur V, Giloni A, Seshadri S (2005). Information sharing in a supply chain under ARMA demand. Management Science, 51(6): 961–969

Giloni A, Hurvich C, Seshadri S (2014). Forecasting and information sharing in supply chains under ARMA demand. IIE Transactions, 46(1): 35–54

Girish M K, Hu J Q (1999). Modeling of correlated arrival processes in the Internet. In: Proceedings of the 38th IEEE Conference on Decision and Control. Phoenix, AZ, 4454–4459

Gong W B, Hu J Q (1992). The MacLaurin series for the GI/G/1 queue. Journal of Applied Probability, 29(1): 176–184

Hu J Q (1996). The departure process of the GI/G/1 queue and its MacLaurin series. Operations Research, 44(5): 810–815

Hu J Q, Zhang C, Zhu C B (2016). (*s, S*) inventory systems with correlated demands. INFORMS Journal on Computing, 28(4): 603–611

Ibragimov R (2009). Copula-based characterizations for higher order Markov processes. Econometric Theory, 25(3): 819–846

Jagerman D L, Melamed B (1992). The transition and autocorrelation structure of TES processes. Communications in Statistics: Stochastic Models, 8(2): 193–219

Jaoua A, L'Ecuyer P, Delorme L (2013). Call-type dependence in multiskill call centers. Simulation, 89(6): 722–734

Kugiumtzis D, Bora-Senta E (2014). Simulation of multivariate non-Gaussian autoregressive time series with given autocovariance and marginals. Simulation Modelling Practice and Theory, 44: 42–53

Kuhl M E, Ivy J S, Lada E K, Steiger N M, Wagner M A, Wilson J R (2010). Univariate input models for stochastic simulation. Journal of Simulation, 4(2): 81–97

Lee H L, Padmanabhan V, Whang S (1997). Information distortion in a supply chain: The bullwhip effect. Management Science, 43(4): 546–558

Lee H L, So K C, Tang C S (2000). The value of information sharing in a two-level supply chain. Management Science, 46(5): 626–643

Lei L, Hu J Q, Zhu C B (2022). Discrete-event stochastic systems with copula correlated input processes. IISE Transactions, 54: 321–331

Lim S Y, Hur S, Noh S J (2006). Departure process of a single server queuing system with Markov renewal input and general service time distribution. Computers & Industrial Engineering, 51(3): 519–525

Livny M, Melamed B, Tsiolis A K (1993). The impact of autocorrelation on queuing systems. Management Science, 39(3): 322–339

Lucantoni D M (1991). New results on the single server queue with a batch Markovian arrival process. Communications in Statistics: Stochastic Models, 7(1): 1–46

Marshall A W, Olkin I (1988). Families of multivariate distributions. Journal of the American Statistical Association, 83(403): 834–841

Melamed B (1991). TES: A class of methods for generating autocorrelated uniform variates. ORSA Journal on Computing, 3(4): 317–329

Melamed B (1993). An overview of TES processes and modeling methodology. In: Donatiello L, Nelson R, eds. Performance Evaluation of Computer and Communication Systems. Berlin, Heidelberg: Springer, 359–393

Melamed B, Hill J R (1995). A survey of TES modeling applications. Simulation, 64(6): 353–370

Nelsen R B (2006). An Introduction to Copulas. 2nd ed. New York, NY: Springer Science & Business Media

Neuts M F (1979). A versatile Markovian point process. Journal of Applied Probability, 16(4): 764–779

Patuwo B E, Disney R L, McNickle D C (1993). The effect of correlated arrivals on queues. IIE Transactions, 25(3): 105–110

Pereira D C, del Rio Vilas D, Monteil N R, Prado R R, del Valle A G (2012). Autocorrelation effects in manufacturing systems performance: A simulation analysis. In: Proceedings of the Winter Simulation Conference (WSC). Berlin: IEEE, 1–12

Runnenburg J T (1962). Some numerical results on waiting-time distributions for dependent arrival-intervals. Statistica Neerlandica, 16(4): 337–347

Shang K H (2012). Single-stage approximations for optimal policies in serial inventory systems with nonstationary demand. Manufacturing & Service Operations Management, 14(3): 414–422

Szekli R, Disney R L, Hur S (1994). MR/GI/1 queues by positively correlated arrival stream. Journal of Applied Probability, 31(2): 497–514

Zhang X (2004). Technical note: Evolution of ARMA demand in supply chains. Manufacturing & Service Operations Management, 6(2): 195–198

Zhu Y, Li H (1993). The MacLaurin expansion for a G/G/1 queue with Markov-modulated arrivals and services. Queuing Systems, 14(1–2): 125–134