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RESEARCH ARTICLE

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Sequential degradation-based burn-in test with multiple periodic inspections

Abstract Burn-in has been proven effective in identifying and removing defective products before they are delivered to customers. Most existing burn-in models adopt a one-shot scheme, which may not be sufficient enough for identification. Borrowing the idea from sequential inspections for remaining useful life prediction and accelerated lifetime test, this study proposes a sequential degradation-based burn-in model with multiple periodic inspections. At each inspection epoch, the posterior probability that a product belongs to a normal one is updated with the inspected degradation level. Based on the degradation level and the updated posterior probability, a product can be disposed, put into field use, or kept in the test till the next inspection epoch. We cast the problem into a partially observed Markov decision process to minimize the expected total burn-in cost of a product, and derive some interesting structures of the optimal policy. Then, algorithms are provided to find the joint optimal inspection period and number of inspections in steps. A numerical study is also provided to illustrate the effectiveness of our proposed model.

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1 Introduction

It is common that a population of products contains a small portion of defective units, because of inherent variability of raw materials, human errors, fluctuation of manufacturing process, etc. These defective products usually exhibit high failure rate or degradation rate. If they are delivered to customers, additional cost may be incurred due to warranty, and the reputation of the company will be affected. Burn-in test has been proven effective in identifying and eliminating these defective products before delivering to customers, such as integrated circuits (Kim, 2011), laser units (Yuan and Ji, 2015), and Micro-Electro-Mechanical Systems (MEMS) devices (Zhai et al., 2016). It can be treated as the final step of manufacturing process, which has been widely adopted in practice (Kuo, 1984).

Many burn-in models have been proposed in the literature. Most of them are developed based on the lifetime distribution of products (Hu et al., 2020a). They assume a bathtub failure rate of products (Jiang and Jardine, 2007; Ye et al., 2011), or a population consists of several subpopulations with different lifetime distribution parameters (Cha, 2011). During the burn-in test, products are run mimicking the real usage patterns, and most often in an accelerated manner. Products that fail before the completion of the test are treated as defective ones, and they are disposed directly. On the other hand, products that survive the test are treated as normal ones, and they are delivered to customers.

However, products are becoming increasingly reliable and experiencing long lifetime with the development of manufacturing technology. The period of a burn-in test is relatively short comparing to the lifetime of a product. Traditional burn-in models that expect defective products fail during the burn-in test become unrealistic (Tseng et al., 2003; Xiang et al., 2013). As a promising solution, degradation-based burn-in tests, promoted by the advance of sensor technology, are attracting more attentions in recent years (Xiang et al., 2012). Specifically, the degradation levels of each product are inspected after the burn-in test. The products whose degradation levels exceed a cutoff line are treated as defective ones and disposed. It is worth noting that Wiener process has been widely adopted to model degradation process, and shows good performance in terms of remaining useful life (RUL) prediction accuracy, see Zhai and Ye (2017), Zhang et al. (2018b), and Hu and Chen (2020), to name a few.

In the degradation-based burn-in test studies stated above, they adopt a one-shot scheme. It may be not adequate for identification, since a normal product has a relatively high probability to be treated as a defective one considering the stochastic property of degradation. Moreover, products whose degradation levels exceed a fixed cutoff line are treated as the same (Zhai et al., 2016). Usually, they have different degradation levels. The degradation level carries information about the product belonging to a normal/defective product. This information may be ignored due to a traditional fixed cutoff line setting.

A sequential multi-inspection degradation-based burn-in test can make full use of these pieces of information to identify normal/defective products. The expected total burn-in cost can then be saved. Indeed, sequential modeling has been widely used in other related areas and proven effective, such as RUL prediction (Lei et al., 2018; Zhang et al., 2018a), and step-stress accelerated degradation test (Tang et al., 2004; Tseng et al., 2009). The RUL prediction accuracy increases significantly as more inspection data are available (Peng et al., 2019; Hu et al. 2020c). Meanwhile, a Bayesian framework has been adopted for designing accelerated degradation test in many studies (Li et al., 2015; Wang et al., 2017).

Based on above motivations, we design a sequential degradation-based burn-in test with multiple inspections in this study. In the test, inspections are conducted periodically to revel the degradation levels of each product. The posterior probability that a product belongs to a normal one is updated with the inspected degradation level. Based on the degradation level and updated posterior probability, the product can be directly disposed, put into field use (i.e., pass the test), or kept in the test till the next inspection epoch. With a given combination of the inspection period and number of inspections of the test, we formulate a partially observed Markov decision process (POMDP) framework to obtain the optimal control policy that minimizes the expected total burn-in cost of a product. Then algorithms are presented to find the joint optimal inspection period and number of inspections in steps.

The remainder of the paper is organized as follows. Section 2 states the model description, where the detail of the test is provided. Section 3 gives an elaborate description of the POMDP framework, and presents a backward induction algorithm and a generalized pattern search (GPS) algorithm to find the joint optimal inspection period and number of inspections in steps. Section 4 provides examples to illustrate the proposed burn-in test model. Section 5 concludes the paper.

2 Model description

2.1 Wiener degradation process

Consider the degradation process of a product, which is modeled by a Wiener process as

$$X(t) = x_0 + \mu t + \sigma B(t), \tag{1}$$

where $x_0 > 0$ is the fixed initial degradation level, $\mu > 0$ and $\sigma > 0$ are the respective drift and diffusion parameters, and $B(\cdot)$ is the standard Brownian motion. Without loss of generality, we assume that the degradation process is generally increasing over time (Liu et al., 2016; Hu et al., 2020b). A degradation failure occurs when the degradation level $X(t)$ first exceeds the predetermined failure threshold X_f. The failure time follows an inverse Gaussian distribution with a probability density function (PDF) $f_\mathrm{IG}(\cdot)$ and a cumulative distribution function (CDF) $F_\mathrm{IG}(\cdot)$ given by

$$f_\mathrm{IG}(t;\ \beta,\ \lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\exp\left[-\frac{\lambda(t-\beta)^2}{2\beta^2 t}\right], \tag{2}$$

$$F_\mathrm{IG}(t;\ \beta,\ \lambda) = \Phi\left[\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\beta}-1\right)\right] + \exp\left(\frac{2\lambda}{\beta}\right)\Phi\left[-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\beta}-1\right)\right], \tag{3}$$

where $\beta = (X_\mathrm{f}-x_0)/\mu$, $\lambda = (X_\mathrm{f}-x_0)^2/\sigma^2$, and Φ is the CDF of the standard Gaussian distribution.

2.2 Two subpopulation assumption

Consider a product population consisting of a normal subpopulation with proportion π_0 and a defective subpopulation with proportion $(1-\pi_0)$. Such a two-subpopulation setting is commonly adopted in burn-in literature, see Jiang and Jardine (2007), Cha and Finkelstein (2010), Ye et al. (2012), and Chen et al. (2020), to name a few. We assume the degradation of the normal class follows a Wiener process with drift parameter $\mu_1 > 0$ and diffusion parameter $\sigma > 0$, while the defective class degrades following a Wiener process with the same σ but a larger drift parameter $\mu_2 > \mu_1$. This assumption indicates that a defective product exhibits a faster degradation speed than a normal one (Zhai et al., 2016).

2.3 Sequential multi-inspection burn-in test

To identify normal products from the population, a sequential burn-in test with multiple periodic inspections is carried out. We assume the inspection period is δ, and the test consists of N inspections. We let π_0 be our belief that a product belongs to the normal class before test, which is given in advance (Zhai et al., 2016). A setup cost c_s is incurred for each product in the burn-in test. Besides, we let the burn-in operation cost be c_o per unit time. At time epoch $t_n = n\delta$ $(n = 1, \dots, N)$, an inspection is carried out to reveal the degradation level with cost c_in. Let π_n be the posterior probability that a product belongs to the normal class when we have observed the degradation data from the burn-in tests at t_n. When the inspected degradation level x_n is revealed, the posterior probability π_n can be updated. Based on (x_n, π_n), we can (i) dispose the product with cost c_d, (ii) keep the product in the burn-in test till the next inspection, or (iii) put it into field use for a mission duration τ with a gain K (e.g., sale price). At the Nth inspection epoch, we have to decide whether to dispose the product or put it into field use. A product is disposed directly with cost c_d if its degradation level exceeds X_f at each inspection epoch. If a product is put into field use but fails before fulfilling the mission duration τ, then a penalty failure cost c_f is incurred. We are trying to find the joint optimal (δ^*, N^*) and the corresponding control policy that minimizes the expected total burn-in cost of a product randomly selected from the population.

3 Optimization

3.1 POMDP framework

In this section, we first fix (δ, N) and cast the optimization problem into a POMDP framework. The resulting optimal burn-in policy $\Pi_n(\delta, N)$ $(n = 1, \dots, N)$ at each inspection epoch and the corresponding expected total burn-in cost $C(\delta, N)$ of a randomly selected product are both function of (δ, N).

The problem is formulated as a POMDP as follows. The state is the type of a randomly selected product (i.e., normal or defective), which is time-constant. The signal is the inspected degradation level at each inspected epoch. The action set includes disposing directly, putting into field use, and keeping in the test till the next inspection epoch. We introduce a value function $V(n, x_n, \pi_n)$ to denote the optimal expected total bun-in cost for the product, where $n = 0, \dots, N$ is the inspection index. Meanwhile, x_n is the degradation level of the product, and π_n is the posterior probability that the product belongs to the normal class given the observed data at time epoch $t_n = n\delta$.

Therefore, the expected total burn-in cost of a randomly selected product is given by

$$C(\delta, N) = V(0, x_0, \pi_0), \tag{4}$$

where x_0 is the initial degradation level of the product, and π_0 is the prior belief that the product belongs to the normal class.

The Bellman equation can be expressed as

$$\begin{aligned} V(n, x_n, \pi_n) = {}& \mathbf{1}_{\{n=1\}} c_\mathrm{s} + \mathbf{1}_{\{n>0\}} (c_\mathrm{o}\delta + c_\mathrm{in}) \\ & + \min\{c_\mathrm{d}, c_\mathrm{u}(x_n, \pi_n), U_\mathrm{b}(n, x_n, \pi_n)\}, \end{aligned} \tag{5}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function that equals 1 if the argument is true and 0 otherwise. Meanwhile, c_s, c_o, c_in, and c_d are costs of a setup, burn-in operation per unit time, an inspection, and disposing a product, respectively. Note that a setup cost is only incurred at the beginning of the test. In Eq. (5), $U_\mathrm{b}(n, x_n, \pi_n)$ is the expected cost of keeping the product in the burn-in test till the next inspection epoch, and $c_\mathrm{u}(x_n, \pi_n)$ is the expected cost of putting the product into field use.

The boundary condition of Eq. (5) is given by

$$V(N, x_N, \pi_N) = c_\mathrm{o}\delta + c_\mathrm{in} + \min\{c_\mathrm{d}, c_\mathrm{u}(x_N, \pi_N)\}. \tag{6}$$

It represents the situation that the burn-in test is complete, in which we have to decide whether to dispose the product or put it into field use. Here, $c_\mathrm{u}(x_N, \pi_N)$ can be expressed as

$$c_\mathrm{u}(x_N, \pi_N) = -K + c_\mathrm{f} F(\tau; x_N, \pi_N), \tag{7}$$

where K is the gain (e.g., sale price) that a product is put into field use, τ is the mission duration that a product is expected to fulfill, and c_f is the penalty cost incurred if a product fails before τ. Meanwhile, $F(\tau; x_N, \pi_N)$ is the probability that the product fails before τ with a starting degradation level x_N and probability of belonging to the normal class π_N, which is given by

$$\begin{aligned} & F(\tau; x_N, \pi_N) \\ & = \pi_N F_\mathrm{IG}(\tau; \beta_1, \lambda) + (1-\pi_N) F_\mathrm{IG}(\tau; \beta_2, \lambda), \end{aligned} \tag{8}$$

where $\beta_i = (X_\mathrm{f} - x_0)/\mu_i$ $(i \in \{1, 2\})$, $\lambda = (X_\mathrm{f} - x_N)^2/\sigma^2$, and $F_\mathrm{IG}(\cdot)$ is the cumulative distribution function of an inverse Gaussian distribution given by Eq. (3).

In Eq. (5), the expected cost of putting the product into field use $c_\mathrm{u}(x_n, \pi_n)$ can refer to Eq. (7). Meanwhile, the expected cost of keeping the product in the test till the $(n+1)$th inspection epoch can be expressed as

$$\begin{aligned} U_\mathrm{b}(n, x_n, \pi_n) = {}& E[V(n+1, \bar{x}_{n+1}, \bar{\pi}_{n+1}) | (x_n, \pi_n)] \\ & + (c_\mathrm{in} + c_\mathrm{o}\delta + c_\mathrm{d}) G(\delta; x_n, \pi_n), \end{aligned} \tag{9}$$

where $G(\delta; x_n, \pi_n)$ is the probability that the degradation level of the product exceeds X_f at the $(n+1)$th inspection epoch. The expression of $G(\delta; x_n, \pi_n)$ is given by

$$G(\delta;\ x_n,\ \pi_n) = \pi_n \int_{X_f}^{+\infty} f_y(\delta \mid \mu_1,\ \sigma,\ x_n) \mathrm{d}y + (1-\pi_n) \int_{X_f}^{+\infty} f_y(\delta \mid \mu_2,\ \sigma,\ x_n) \mathrm{d}y, \quad (10)$$

where $f_y(\delta \mid \mu_i,\ \sigma,\ x_n)$ is PDF of the degradation level of a product after time duration δ with drift parameter μ_i and diffusion parameter σ, conditional on that the starting degradation level is x_n, which can be expressed as

$$f_y(\delta \mid \mu_i,\ \sigma,\ x_n) = \frac{1}{\sqrt{2\pi\sigma^2\delta}} \exp\left[-\frac{(y-x_n-\mu_i\delta)^2}{2\sigma^2\delta}\right]. \quad (11)$$

Moreover, in Eq. (9), $E[V(n+1,\ \overline{x}_{n+1},\ \overline{\pi}_{n+1}) \mid (x_n,\ \pi_n)]$ is the expected cost to go given that the product degradation level is less than X_f at the $(n+1)$th inspection epoch, which can be expressed as

$$E[V(n+1,\ \overline{x}_{n+1},\ \overline{\pi}_{n+1}) \mid (x_n,\ \pi_n)]$$
$$= \pi_n \int_{-\infty}^{X_f} f_y(\delta \mid \mu_1,\ \sigma,\ x_n) V\left(n+1,\ y,\ \pi_{n+1}^y(x_n,\ \pi_n)\right) \mathrm{d}y + (1-\pi_n) \int_{-\infty}^{X_f} f_y(\delta \mid \mu_2,\ \sigma,\ x_n) V\left(n+1,\ y,\ \pi_{n+1}^y(x_n,\ \pi_n)\right) \mathrm{d}y, \quad (12)$$

where $\pi_{n+1}^y(x_n,\ \pi_n)$ is the updated posterior probability that the product belongs to the normal class when observing y at epoch $(n+1)\delta$, given that the state at epoch n is $(x_n,\ \pi_n)$. Given that the probability of the product belonging to the normal class is π_n and the degradation level is x_n at the nth inspection epoch, the probability density that the degradation level is y at the $(n+1)$th epoch is

$$f\left(x_{n+1} = y \mid (x_n,\ \pi_n)\right) = \pi_n f_y(\delta \mid \mu_1,\ \sigma,\ x_n) + (1-\pi_n) f_y(\delta \mid \mu_2,\ \sigma,\ x_n). \quad (13)$$

Therefore, $\pi_{n+1}^y(x_n,\ \pi_n)$ can be updated based on Bayes' rule as

$$\pi_{n+1}^y(x_n,\ \pi_n) = \frac{\pi_n f_y(\delta \mid \mu_1,\ \sigma,\ x_n)}{f\left(x_{n+1} = y \mid (x_n,\ \pi_n)\right)} = \frac{\pi_n f_y(\delta \mid \mu_1,\ \sigma,\ x_n)}{\pi_n f_y(\delta \mid \mu_1,\ \sigma,\ x_n) + (1-\pi_n) f_y(\delta \mid \mu_2,\ \sigma,\ x_n)}. \quad (14)$$

This finalizes our POMDP formulation. We will investigate the structure of the optimal policy and provide an algorithm to obtain it based on this framework in the next subsection.

3.2 Analysis and algorithms for the POMDP

Regarding our POMDP framework, we first have following two important propositions.

Proposition 1. Given n and π_n, both $U_b(n,\ x_n,\ \pi_n)$ and $c_u(x_n,\ \pi_n)$ are non-decreasing with x_n, and the optimal policy is a control-limit policy with respect to x_n. If $c_u(x_0,\ \pi_n) < \min\{U_b(n,\ x_0,\ \pi_n),\ c_d\}$, there exist two thresholds $x_0 < \overline{x}_1 < \overline{x}_2 < +\infty$ that the optimal actions are putting into field use, keeping in the test till the next inspection epoch, or disposing directly when $x_n < \overline{x}_1$, $\overline{x}_1 \leqslant x_n < \overline{x}_2$, and $x_n \geqslant \overline{x}_2$, respectively. Moreover, $V(n,\ x_n,\ \pi_n)$ is non-decreasing with x_n.

Proposition 2. Given n and x_n, both $U_b(n,\ x_n,\ \pi_n)$ and $c_u(x_n,\ \pi_n)$ are non-increasing with π_n, and the optimal policy is a control-limit policy with respect to π_n. If $c_u(x_n,\ 1) < \min\{U_b(n,\ x_n,\ 1),\ c_d\}$, there exist two thresholds $0 < \overline{\pi}_1 < \overline{\pi}_2 < 1$ that the optimal actions are putting into field use, keeping in the test till the next inspection epoch, or disposing directly when $\pi_n \geqslant \overline{\pi}_2$, $\overline{\pi}_1 \leqslant \pi_n < \overline{\pi}_2$, and $\pi_n < \overline{\pi}_1$, respectively. Moreover, $V(n,\ x_n,\ \pi_n)$ is non-increasing with π_n.

The proofs of Propositions 1 and 2 are given in Appendices A.1 and A.2, respectively. Then, another two important propositions are deduced as follows.

Proposition 3. At any inspection epoch n, if the degradation level x_n satisfies $x_n \geqslant \overline{X}$, where $F\left(\tau,\ \overline{X},\ 1\right) = (c_d + K)/c_f$, then the optimal policy is disposing the product directly, regardless of the value of π_n.

Proposition 4. If there exists a $\overline{\Theta} = \max\left\{\overline{\Theta}_1,\ \overline{\Theta}_2\right\}$ such that $F\left(\tau,\ x_0,\ \overline{\Theta}_1\right) = (c_s + c_o\delta + c_{in})/c_f$ and $F\left(\tau,\ x_0,\ \overline{\Theta}_2\right) = (c_d + K)/c_f$, then the optimal policy is putting the product into field use without carrying out a burn-in test when $\pi_0 \geqslant \overline{\Theta}$.

The proofs of Propositions 3 and 4 are given in Appendices A.3 and A.4, respectively. Proposition 3 indicates that if the degradation level of a product exceeds a threshold at an inspection epoch, then it should be disposed directly without calculating Eq. (5). This proposition is useful for saving the time of obtaining the optimal policy. Proposition 4 represents the situation in which π_0 is sufficiently large that the expected cost of putting a randomly selected product into field use is smaller than the cost of conducting a burn-in test. Then it is not necessary to conduct a burn-in test.

For a fixed $(\delta,\ N)$, we present a backward induction algorithm in **Algorithm 1** to find the optimal burn-in policy $\prod_n(\delta,\ N)$ and the corresponding $C(\delta,\ N)$.

Algorithm 1 Backward induction algorithm to calculate $\Pi_n(\delta, N)$ and $C(\delta, N)$

Input: $\pi_0, \delta, N, c_{\text{in}}, c_{\text{s}}, c_{\text{o}}, c_{\text{d}}, c_{\text{f}}, K, X_{\text{f}}, \mu_1, \mu_2, x_0, \tau, N_x, N_\pi$
Output: The optimal action at each state $\Pi_n(\delta, N)$ and the corresponding $C(\delta, N)$

```
begin
  Discretize the state space by introducing two positive integers N_x and N_π as scaling parameters
   and denoting δ_x = X_f/N_x and δ_π = π/N_π as the granularity for discretization, respectively
  1. for x_N = x_0 : δ_x : X_f do
       for π_N = 0 : δ_π : π do
         Calculate V(N, x_N, π_N) based on Eq. (6)
       end
     end
  2. for n = 0 : N do
       for x_n = x_0 : δ_x : X_f do
         for π_n = 0 : δ_π : π do
           Calculate V(n, x_n, π_n) based on Eq. (5)
         end
       end
     end
  3. Find the optimal action at each state Π_n(δ, N)
  4. Obtain the C(δ, N) = V(0, x_0, π_0)
end
```

3.3 GPS algorithm and test flowchart

In this section, we first present an algorithm to find the optimal (δ^*, N^*) that minimizes $C(\delta, N)$. Numerical optimization methods, such as Newton's method and trust-region method, are not applicable here, because a closed-form expression of $C(\delta, N)$ is not available. The GPS algorithm (Audet and Dennis Jr, 2002) is thus adopted to find the optimal (δ^*, N^*), which is shown to be effective in searching the optimal value of nonlinear, non-smooth, or even non-continuous functions (Huynh et al., 2014; Sun et al., 2020). Given the initial guess and a parameter Λ_0, the algorithm iterates as follows:

• At the ith iteration, based on **Algorithm 1**, calculate the $C(\delta, N)$ at some trial points on a mesh centered at the current optimal solution, where the mesh size is proportional to the parameter Λ_i;

• If any of the trial points has a smaller $C(\delta, N)$ than the current optimal solution, we set this point as a new current optimal solution and increase the mesh size parameter such that $\Lambda_{i+1} \geqslant \Lambda_i$;

• Otherwise, we stay at the current optimal solution and decrease the mesh size at the next step so that $\Lambda_{i+1} < \Lambda_i$.

The above steps iterate until the mesh size parameter Λ is smaller than a predetermined threshold.

In practice, we obtain the optimal (δ^*, N^*) and the corresponding control policy $\prod_n(\delta^*, N^*)$ based on the algorithm stated above. The flowchart of the test for each unit is illustrated in Fig. 1. At the nth $(n < N^*)$ inspection epoch, the posterior probability that the unit belongs to the normal class is updated with the inspected degradation level. Then the optimal action (i.e., dispose, keep in the test, or put into field use) is implemented based on the degradation level and posterior probability according to the control policy $\prod_n(\delta^*, N^*)$. At the N^*th inspection epoch, the optimal action (i.e., dispose or put into field use) is implemented based on the degradation level and the updated posterior probability according to the control policy $\prod_n(\delta^*, N^*)$, and the test stops.

4 Illustrative example

4.1 Numerical study

In this section, we conduct a numerical study to illustrate our proposed model. The proportion of normal products is $\pi_0 = 0.8$. We set the drift parameters of normal and defective products to be, respectively, $\mu_1 = 2$ and $\mu_2 = 4$. The diffusion parameter is $\sigma = 1$. The initial degradation level is $x_0 = 0$. The costs of a setup, an inspection, burn-in operation cost per unit, disposing a product, and failing before mission duration reached are set to be $c_{\text{s}} = 1$, $c_{\text{in}} = 3$, $c_{\text{o}} = 0.3$, $c_{\text{d}} = 30$, and $c_{\text{f}} = 390$, respectively. Gain of a product putting into field use is $K = 100$. The degradation failure threshold is $x_{\text{f}} = 80$. The mission duration is $\tau = 35$.

Based on the algorithms provided in Sections 3.2 and

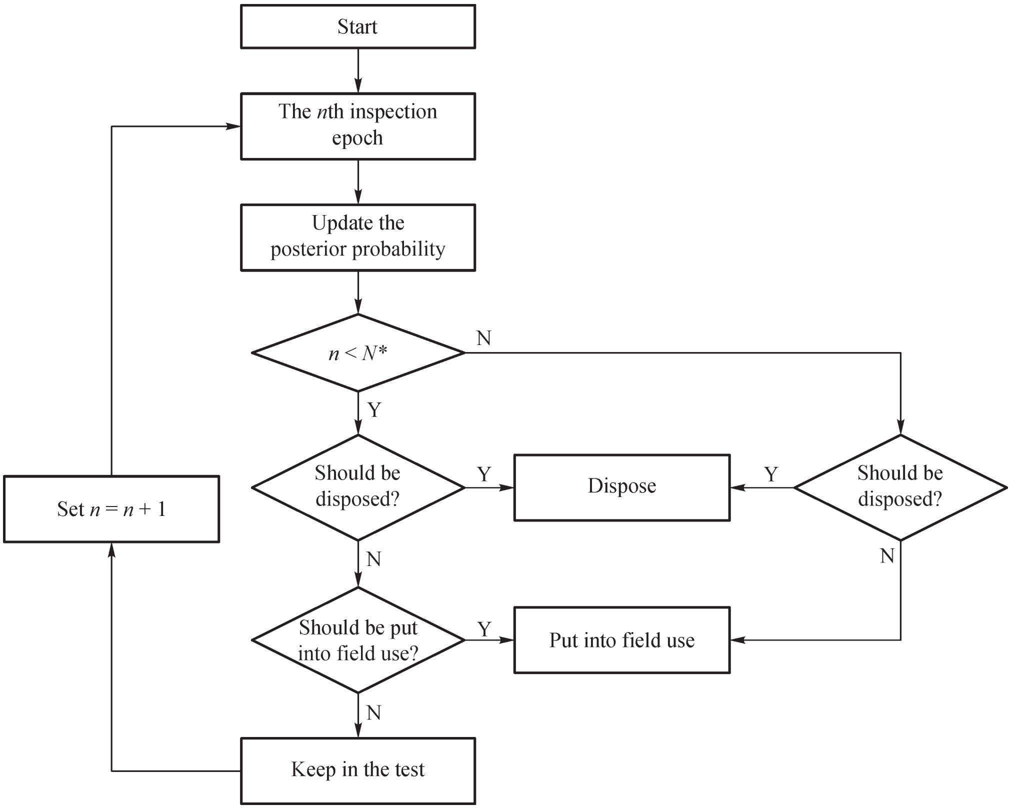

Fig. 1 Flowchart of the proposed burn-in test.

3.3, we find that $(\delta^*, N^*) = (0.7451, 4)$ achieves the minimal expected total burn-in cost of a randomly selected product as $C = -28.3120$. The optimal policy $\Pi(0.7451, 4)$ regarding n, x_n, and π_n is illustrated in Fig. 2. In Fig. 2, the optimal actions at $(x_n \geqslant 8, \forall \pi_n)$ are disposing the product, which is not displayed. The result validates that the optimal policy is a control-limit policy with respect to x_n and π_n at each inspection epoch. From Figs. 2a, 2b and 2c, we can see that given a $x_n \leqslant 6$, the optimal action changes from disposing directly, to keep in the test till next inspection, then to put into field use as π_n increases. Meanwhile, we can see that given $\pi_n = 0.9$, the optimal action changes from disposing directly, to keep in the test till next inspection, then to put into field use as x_n decreases. The goal of our test is to identify the normal one in terms of the posterior probability, while a product can be directly disposed when the degradation level exceeds a threshold. It coincides with Proposition 3.

We then consider two one-shot scheme models for comparison.

• Model OS (one shot based on a same cutoff line): The duration of the burn-in test is $T_{\rm OS}$, and a product is put into field use if its degradation level is less than $X_{\rm OS}$ after the test; otherwise, it is disposed (Zhai et al., 2016).

• Model OP (one shot based on the posterior probability): Only one inspection with period $T_{\rm OP}$ is executed for our proposed model.

We can find $(T^*_{\rm OS}, X^*_{\rm OS}) = (0.9708, 3.0191)$ for Model OS, and the corresponding expected total burn-in cost of a randomly selected product is $C_{\rm OS} = -21.8842$. Meanwhile, we can obtain $T^*_{\rm OP} = 1.1395$ for Model OP, and the corresponding expected total burn-in cost of a randomly selected product is $C_{\rm OP} = -23.7110$. Comparing Model OP to Model OS, one can see that, using the posterior probability rather than a same cutoff line makes use of the degradation levels of each product, and achieves a lower expected total burn-in cost. Further, when sequential multiple inspections are executed, a lower expected total burn-in cost $C = -28.3120$ is achieved with $n = 4$. It validates the effectiveness of our proposed model.

We further conduct a sensitivity study to see how the optimal policy varies with some critical input parameters. Note that when one parameter varies, other parameters remain fixed as the base case stated at the beginning of Section 4.1. First, we consider the variation of the proportion of normal products π_0. The results of the proposed model, Model OS, and Model OP are listed in Table 1. One can see that the expected total burn-in cost of a randomly selected product of all the three models increases as π_0 decreases. It is intuitive that more cost will be incurred with a higher proportion of defective products in a population. Moreover, the inspection period of all the models increases as π_0 decreases. In Table 1, NA indicates that it is not cost-effective to conduct a burn-in test under

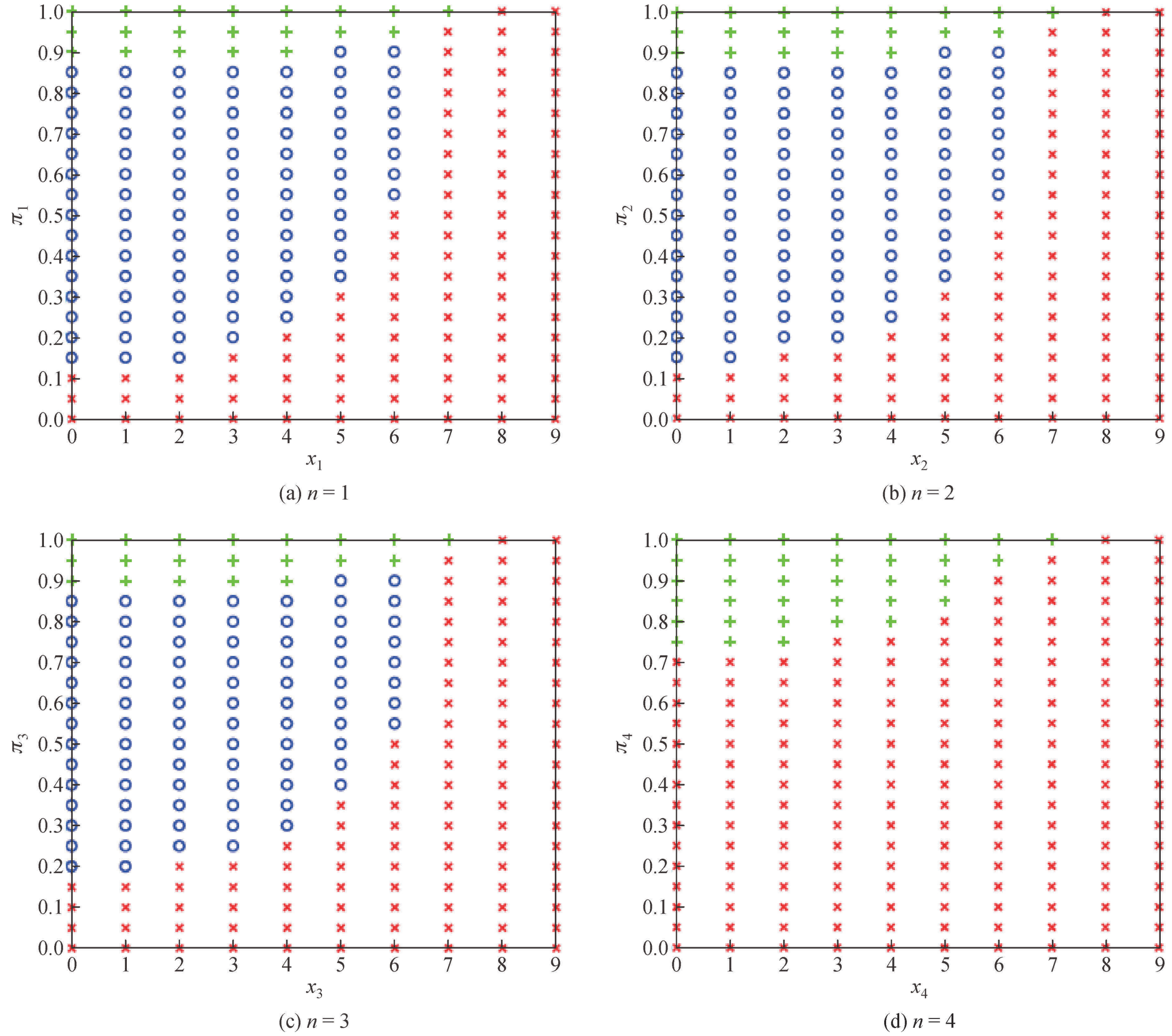

Fig. 2 Optimal action at each inspection epoch (×, O, and + indicate "dispose", "keep in the test", and "put into field use", respectively).

the current parameter setting, which meets the situation indicated in Proposition 4. Therefore, the costs of setup and inspection should be decreased to make the burn-in test more cost-effective. It is worth noting that our proposed model outperforms the other two models in terms of a smaller expected total burn-in cost of a randomly selected product.

We then consider the variation of the drift parameter of a normal product μ_1. The results of the proposed model, Model OS, and Model OP are listed in Table 2. One can see, the expected total burn-in cost of a randomly selected product of all the three models decreases as μ_1 decreases. It is because that the difference of degradation rates between a normal product and a defective product increases as μ_1 decreases and μ_2 remains fixed. Then cost can be saved to identify the normal ones. Besides, the failure risk of a normal product after being delivered to customer decreases. It saves the expected total cost significantly. Meanwhile, the inspection period increases accordingly. It is beneficial to differentiate a normal one from a defective one with a longer period. On the other hand, the expected total burn-in cost increases as μ_1 increases, because the difference of degradation rates between these two types of products decreases. Besides, the failure risk of a normal product after being delivered to customer increases, resulting in a higher expected total burn-in cost.

Finally, we consider the variation of the diffusion parameter σ. The results of the proposed model, Model OS, and Model OP are listed in Table 3. One can see, the expected total burn-in cost of a randomly selected product of all the three models increases as σ increases. It is because that the stochastic property increases with σ, which increases the difficulty of identification. Then the expected total burn-in cost increases. Meanwhile, the inspection periods of proposed model and Model OP increase as σ increases. However, the inspection period of Model OS decreases as σ increases.

It is worth noting that our proposed model achieves the

Table 1 Sensitivity study on π_0

π_0	Proposed model		Model OS		Model OP	
	$(\delta^*,\ N^*)$	C	$(T_{\mathrm{OS}}^*,\ X_{\mathrm{OS}}^*)$	C_{OS}	T_{OP}	C_{OP}
0.7	(0.8235, 4)	−16.1169	(1.1916, 3.3791)	−10.3925	1.4135	−13.0565
0.8	(0.7451, 4)	−28.3120	(0.9708, 3.0191)	−21.8842	1.1395	−23.7110
0.9	NA	−43.7135	(0.4829, 2.0321)	−38.8970	NA	−43.7135

Table 2 Sensitivity study on μ_1

μ_1	Proposed model		Model OS		Model OP	
	$(\delta^*,\ N^*)$	C	$(T_{\mathrm{OS}}^*,\ X_{\mathrm{OS}}^*)$	C_{OS}	T_{OP}	C_{OP}
1.0	(5.9782, 4)	−67.6575	(3.8309, 9.8084)	−68.6076	6.8135	−67.6110
2.0	(0.7451, 4)	−28.3120	(0.9708, 3.0191)	−21.8842	1.1395	−23.7110
2.2	(0.3023, 4)	28.7240	(0.5125, 4.0231)	33.7341	0.3245	29.7630

Table 3 Sensitivity study on σ

σ	Proposed model		Model OS		Model OP	
	$(\delta^*,\ N^*)$	C	$(T_{\mathrm{OS}}^*,\ X_{\mathrm{OS}}^*)$	C_{OS}	T_{OP}	C_{OP}
0.5	(0.6353, 4)	−66.0352	(1.2538, 3.8409)	−65.8514	0.6532	−62.2863
1.0	(0.7451, 4)	−28.3120	(0.9708, 3.0191)	−21.8842	1.1395	−23.7110
1.5	(0.8011, 3)	9.3966	(0.9215, 2.4026)	11.1035	1.2125	11.0775

lowest expected total burn-in cost of a randomly selected product in almost all the cases in the sensitivity study. It again validates the effectiveness of our proposed model.

4.2 Case study

In this section, we provide a MEMS device example to illustrate the proposed model (Peng et al., 2009; Zhai et al., 2016). The micro-engine inside the MEMS suffers degradation type failure. Its population consists of two subpopulations, and the proportion of normal ones is $\pi_0 = 0.95$. The degradation can be modeled by the Wiener process, where the drift parameters of normal and defective products are, respectively, $\mu_1 = 8.4832 \times 10^{-9}$ and $\mu_2 = 2.6875 \times 10^{-8}$. The diffusion parameters are the same, which is $\sigma = 2.2808 \times 10^{-8}$. The designed lifetime of a product is $\tau = 1200$ units. The failure threshold is $X_{\mathrm{f}} = 1.3093 \times 10^{-5}$. The costs of a setup, an inspection, burn-in operation cost per unit, disposing a product, and failing before mission duration reached are set to be $c_{\mathrm{s}} = 1$, $c_{\mathrm{in}} = 0.5$, $c_{\mathrm{o}} = 0.08$, $c_{\mathrm{d}} = 30$, and $c_{\mathrm{f}} = 390$, respectively. Gain of a product putting into field use is $K = 100$.

Based on the algorithms provided in Sections 3.2 and 3.3, we can find that $(\delta^*,\ N^*) = (57.11,\ 2)$ achieves the minimal expected total burn-in cost of a randomly selected product as $C = -88.72$. The optimal policy $\Pi(57.11,\ 2)$ regarding n, x_n, and π_n is illustrated in Fig. 3. We take the Model OS presented in Zhai et al. (2016) for comparison, where $(T_{\mathrm{OS}}^*,\ X_{\mathrm{OS}}^*) = (68.49,\ 1.35 \times 10^{-6})$ and the corresponding expected total burn-in cost of a randomly selected product is $C_{\mathrm{OS}} = -86.38$. It again validates the effectiveness of our proposed model.

5 Conclusions

This study proposes a sequential degradation-based burn-in model with multiple periodic inspections. Based on the inspected degradation level, the posterior probability that a product belongs to the normal class is updated. Then a product can be directly disposed, put into field use, or kept in the test till the next inspection based on the degradation level and the updated posterior probability. With a given combination of inspection period and number of inspections, we first formulate a POMDP framework to obtain the optimal policy that minimizes the expected total burn-in cost of a randomly selected product. We prove that the optimal policy is a control-limit policy with respect to both the degradation level and the posterior probability. We then provide algorithms to find the joint optimal inspection period and number of inspections that minimizes the expected total burn-in cost in steps. We also provide a numerical study to illustrate our proposed model. The results demonstrate that our proposed method is capable of identifying defective products with a lower expected cost per unit, comparing with the classic one-shot scheme method.

In this study, we assume that the test stops when the number of inspections reaches a value, which is set as one of the decision variables. If we assume that the test stops

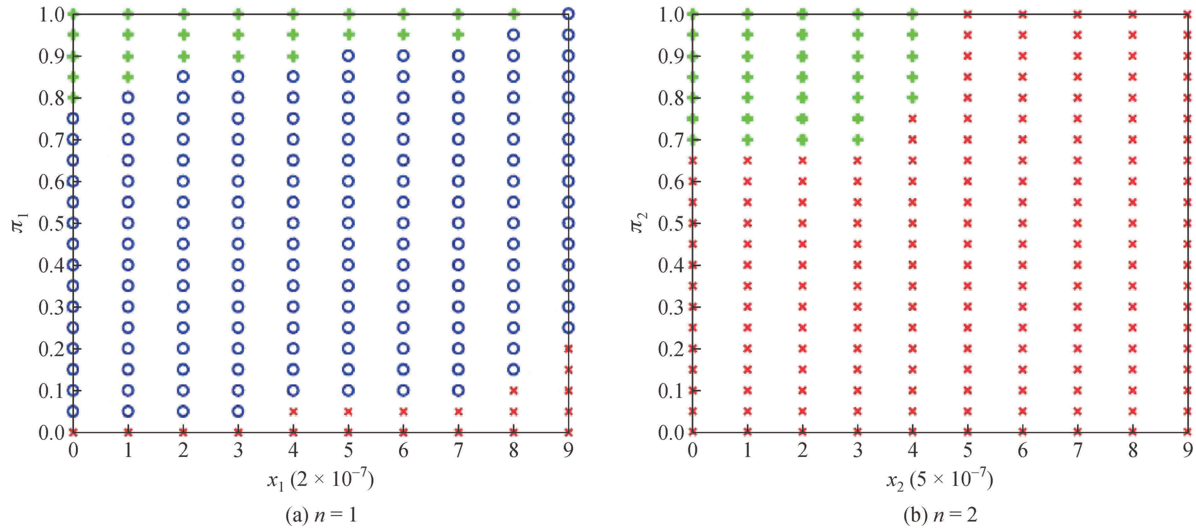

Fig. 3 Optimal action at each inspection epoch (×, ○, and + indicate "dispose", "keep in the test", and "put into field use", respectively).

when the posterior probability becomes large enough, then the test may be endless. We set that the penalty cost is fixed in this study. In some applications, it may vary with age, which is worth investigating in future. Meanwhile, dynamic inspection interval has been adopted in RUL prediction and condition-based maintenance (Grall et al., 2002; Ponchet et al., 2010; Huynh, 2012). It is interesting to investigate a burn-in policy with multiple inspections where the inspection interval depends on the current degradation level.

Appendix A

A.1 Proof of Proposition 1

1) We prove that $U_{\rm b}(n, x_n, \pi_n)$ is non-decreasing with x_n given n and π_n.

We prove it by induction. We first show that $V(N, x_N, \pi_N)$ is non-decreasing with x_N. We have

$$c_{\rm u}(x_N, \pi_N) = -K + c_{\rm f}[\pi_N F_{\rm IG}(\tau; \beta_1, \lambda) + (1-\pi_N)F_{\rm IG}(\tau; \beta_2, \lambda)].$$

It is clear that $c_{\rm u}(x_N, \pi_N)$ increases with x_N since $F_{\rm IG}(\cdot)$ increases with x_N. Recall that

$$V(N, x_N, \pi_N) = c_{\rm o}\delta + c_{\rm in} + \min\{c_{\rm d}, c_{\rm u}(x_N, \pi_N)\},$$

then we have $V(N, x_N, \pi_N)$ is non-decreasing with x_N, and $\max\limits_{x_N} V(N, x_N, \pi_N) = V_{\max} = c_{\rm o}\delta + c_{\rm in} + c_{\rm d}$.

Second, we assume that $V(n+1, x_{n+1}, \pi_{n+1})$ is non-decreasing with x_{n+1}, and $\max\limits_{x_{n+1}} V(N, x_{n+1}, \pi_{n+1}) = V_{\max}$. We let $x_{n1} > x_{n2}$, and let $f_y(\mu_i, x_{nj})$ denotes $f_y(\delta \mid \mu_i, \sigma, x_{nj})$ for simplicity, then we have

$$\begin{aligned}
&U_{\rm b}(n, x_{n1}, \pi_n) - U_{\rm b}(n, x_{n2}, \pi_n)\\
&= \pi_n\left\{\int_{-\infty}^{X_{\rm f}} f_y(\mu_1, x_{n1})V\left(n+1, y, \pi_{n+1}^{y}(x_{n1}, \pi_n)\right){\rm d}y + \int_{X_{\rm f}}^{+\infty} f_y(\mu_1, x_{n1})V_{\max}{\rm d}y\right\}\\
&\quad -\pi_n\left\{\int_{-\infty}^{X_{\rm f}} f_y(\mu_1, x_{n2})V\left(n+1, y, \pi_{n+1}^{y}(x_{n2}, \pi_n)\right){\rm d}y + \int_{X_{\rm f}}^{+\infty} f_y(\mu_1, x_{n2})V_{\max}{\rm d}y\right\}\\
&\quad + (1-\pi_n)\left\{\int_{-\infty}^{X_{\rm f}} f_y(\mu_2, x_{n1})V\left(n+1, y, \pi_{n+1}^{y}(x_{n1}, \pi_n)\right){\rm d}y + \int_{X_{\rm f}}^{+\infty} f_y(\mu_2, x_{n1})V_{\max}{\rm d}y\right\}\\
&\quad -(1-\pi_n)\left\{\int_{-\infty}^{X_{\rm f}} f_y(\mu_2, x_{n2})V\left(n+1, y, \pi_{n+1}^{y}(x_{n2}, \pi_n)\right){\rm d}y + \int_{X_{\rm f}}^{+\infty} f_y(\mu_2, x_{n2})V_{\max}{\rm d}y\right\}.
\end{aligned}$$

Here, we only need to prove

$$\begin{aligned}
\Omega = &\int_{-\infty}^{X_{\rm f}} f_y(\mu_1, x_{n1})V\left(n+1, y, \pi_{n+1}^{y}(x_{n1}, \pi_n)\right){\rm d}y + \int_{X_{\rm f}}^{+\infty} f_y(\mu_1, x_{n1})V_{\max}{\rm d}y\\
&-\int_{-\infty}^{X_{\rm f}} f_y(\mu_1, x_{n2})V\left(n+1, y, \pi_{n+1}^{y}(x_{n2}, \pi_n)\right){\rm d}y - \int_{X_{\rm f}}^{+\infty} f_y(\mu_1, x_{n2})V_{\max}{\rm d}y \geqslant 0.
\end{aligned}$$

Note that $f_y(\mu_1,\ x_{n1}) = f_{y+x_{n1}-x_{n2}}(\mu_1,\ x_{n2})$ and $f_y(\mu_2,\ x_{n1}) = f_{y+x_{n1}-x_{n2}}(\mu_2,\ x_{n2})$. Therefore, we have

$$\Omega = \int_{-\infty}^{X_\mathrm{f}-x_{n1}+x_{n2}} f_y(\mu_1,\ x_{n1})\left[V\left(n+1,\ y+x_{n1}-x_{n2},\ \pi_{n+1}^{y+x_{n1}-x_{n2}}(x_{n2},\ \pi_n)\right)-V\left(n+1,\ y,\ \pi_{n+1}^{y}(x_{n1},\ \pi_n)\right)\right]\mathrm{d}y$$
$$+\int_{X_\mathrm{f}-x_{n1}-x_{n2}}^{X_\mathrm{f}} f_y(\mu_1,\ x_{n2})\left[V_{\max}-V\left(n+1,\ y,\ \pi_{n+1}^{y}(x_{n2},\ \pi_n)\right)\right]\mathrm{d}y \geqslant 0,$$

since $V(n+1,\ x_{n+1},\ \pi_{n+1})$ is non-decreasing with x_{n+1}, and $\max\limits_{x_{n+1}} V(N,\ x_{n+1},\ \pi_{n+1}) = V_{\max}$.

2) Same as $c_\mathrm{u}(x_N,\ \pi_N)$, we prove that $c_\mathrm{u}(x_n,\ \pi_n)$ increases with x_n given n and π_n.

3) Recall that

$$V(n,\ x_n,\ \pi_n) = \mathbf{1}_{\{n=1\}}c_\mathrm{s} + \mathbf{1}_{\{n>0\}}(c_\mathrm{o}\delta + c_\mathrm{in}) + \min\{c_\mathrm{d},\ c_\mathrm{u}(x_n,\ \pi_n),\ U_\mathrm{b}(n,\ x_n,\ \pi_n)\},$$

and both $U_\mathrm{b}(n,\ x_n,\ \pi_n)$ and $c_\mathrm{u}(x_n,\ \pi_n)$ is non-decreasing with x_n given n and π_n. Moreover, $\max\limits_{x_n}\{U_\mathrm{b}(n,\ x_n,\ \pi_n),\ c_\mathrm{u}(x_n,\ \pi_n)\} \leqslant \mathbf{1}_{\{n=1\}}c_\mathrm{s} + V_{\max}$.

Therefore, given n and π_n, the optimal policy is a control-limit policy with respect to x_n, and $V(n,\ x_n,\ \pi_n)$ is non-decreasing with x_n.

Meanwhile, if $c_\mathrm{u}(x_0,\ \pi_n) < \min\{U_\mathrm{b}(n,\ x_0,\ \pi_n),\ c_\mathrm{d}\}$, there exist two thresholds $x_0 < \overline{x}_1 < \overline{x}_2 < +\infty$ that the optimal actions are putting into field use, keeping in the test till the next inspection epoch, or disposing directly when $x_n < \overline{x}_1$, $\overline{x}_1 \leqslant x_n < \overline{x}_2$, and $x_n \geqslant \overline{x}_2$, respectively. This finishes the proof.

A.2 Proof of Proposition 2

1) We prove that $U_\mathrm{b}(n,\ x_n,\ \pi_n)$ is non-increasing with π_n given n and x_n.

We prove it by induction. We first show that $V(N,\ x_N,\ \pi_N)$ is non-increasing with π_N. It is clear that $c_\mathrm{u}(x_N,\ \pi_N)$ decreases with π_N since $F_\mathrm{IG}(\tau;\ \beta_1,\ \lambda) < F_\mathrm{IG}(\tau;\ \beta_2,\ \lambda)$. Recall that

$$V(N,\ x_N,\ \pi_N) = c_\mathrm{o}\delta + c_\mathrm{in} + \min\{c_\mathrm{d},\ c_\mathrm{u}(x_N,\ \pi_N)\},$$

then we have $V(N,\ x_N,\ \pi_N)$ is non-increasing with π_N, and $\max\limits_{\pi_N} V(N,\ x_N,\ \pi_N) = V_{\max}$.

Second, we assume that $V(n+1,\ x_{n+1},\ \pi_{n+1})$ is non-increasing with π_{n+1}, and $\max\limits_{\pi_{n+1}} V(n+1,\ x_{n+1},\ \pi_{n+1}) = V_{\max}$. We let $f_y(\mu_i)$ denote $f_y(\delta\mid\mu_i,\ \sigma,\ x_n)$ for simplicity, then we have

$$U_\mathrm{b}(n,\ x_n,\ \pi_n) = \pi_n\left\{\begin{array}{l}\displaystyle\int_{-\infty}^{X_\mathrm{f}} f_y(\mu_1)V\left(n+1,\ y,\ \pi_{n+1}^{y}(x_n,\ \pi_n)\right)\mathrm{d}y + \int_{X_\mathrm{f}}^{+\infty} f_y(\mu_1)V_{\max}\mathrm{d}y \\ \displaystyle-\int_{-\infty}^{X_\mathrm{f}} f_y(\mu_2)V\left(n+1,\ y,\ \pi_{n+1}^{y}(x_n,\ \pi_n)\right)\mathrm{d}y - \int_{X_\mathrm{f}}^{+\infty} f_y(\mu_2)V_{\max}\mathrm{d}y\end{array}\right\} + C,$$

where C is a constant. Note that $\pi_{n+1}^{y}(x_n,\ \pi_n)$ increases with π_n, and $V(n+1,\ x_{n+1},\ \pi_{n+1})$ is non-increasing with π_{n+1}. Therefore, it is clear that $U_\mathrm{b}(n,\ x_n,\ \pi_n)$ is non-increasing with π_n.

2) Same as $c_\mathrm{u}(x_N,\ \pi_N)$, we prove that $c_\mathrm{u}(x_n,\ \pi_n)$ decreases with π_n given n and x_n.

3) Recall that

$$V(n,\ x_n,\ \pi_n) = \mathbf{1}_{\{n=1\}}c_\mathrm{s} + \mathbf{1}_{\{n>0\}}(c_\mathrm{o}\delta + c_\mathrm{in}) + \min\{c_\mathrm{d},\ c_\mathrm{u}(x_n,\ \pi_n),\ U_\mathrm{b}(n,\ x_n,\ \pi_n)\},$$

and both $U_\mathrm{b}(n,\ x_n,\ \pi_n)$ and $c_\mathrm{u}(x_n,\ \pi_n)$ is non-increasing with π_n given n and x_n. Moreover, $\max\limits_{\pi_n}\{U_\mathrm{b}(n,\ x_n,\ \pi_n),\ c_\mathrm{u}(x_n,\ \pi_n)\} \leqslant \mathbf{1}_{\{n=1\}}c_\mathrm{s} + V_{\max}$.

Therefore, given n and x_n, the optimal policy is a control-limit policy with respect to π_n, and $V(n,\ x_n,\ \pi_n)$ is non-increasing with π_n.

Meanwhile, if $c_\mathrm{u}(x_n,\ 1) < \min\{U_\mathrm{b}(n,\ x_n,\ 1),\ c_\mathrm{d}\}$, there exist two thresholds $0 < \overline{\pi}_1 < \overline{\pi}_2 < 1$ that the optimal actions are putting into field use, keeping in the test till the next inspection epoch, or disposing directly when $\pi_n \geqslant \overline{\pi}_2$, $\overline{\pi}_1 \leqslant \pi_n < \overline{\pi}_2$, and $\pi_n < \overline{\pi}_1$, respectively. This finishes the proof.

A.3 Proof of Proposition 3

Based on Propositions 1 and 2, we have $c_\mathrm{u}(x_n,\ \pi_n)$ is non-increasing with π_n, while increasing with x_n. Then we have $c_\mathrm{d} < c_\mathrm{u}(x_N,\ \pi_N)$, since $-K + c_\mathrm{f}F\left(\tau,\ \overline{X},\ 1\right) = c_\mathrm{d}$, and $x_N > \overline{X}$. We have

$$V(N,\ x_N,\ \pi_N) = c_\mathrm{o}\delta + c_\mathrm{in} + c_\mathrm{d}.$$

Then we assume $V(n+1,\ x_{n+1},\ \pi_{n+1}) = c_\mathrm{o}\delta + c_\mathrm{in} + c_\mathrm{d}$. We therefore have

$$U_\mathrm{b}(n,\ x_n,\ \pi_n) = V(n+1,\ x_{n+1},\ \pi_{n+1}) = c_\mathrm{o}\delta + c_\mathrm{in} + c_\mathrm{d},$$

since we assume the degradation process is generally increasing over time.

Then we have

$$c_{\mathrm{d}} \leqslant \min\{c_{\mathrm{u}}(x_n, \pi_n), U_{\mathrm{b}}(n, x_n, \pi_n)\}, \forall x_n > \overline{X}.$$

This finishes the proof.

A.4 Proof of Proposition 4

Based on Propositions 1 and 2, we have both $U_{\mathrm{b}}(n, x_n, \pi_n)$ and $c_{\mathrm{u}}(x_n, \pi_n)$ are non-increasing with π_n and non-decreasing with x_n. Recall that

$$V(1, x_1, \pi_1) = c_{\mathrm{s}} + c_{\mathrm{o}}\delta + c_{\mathrm{in}} + \min\{c_{\mathrm{d}}, c_{\mathrm{u}}(x_1, \pi_1), U_{\mathrm{b}}(1, x_1, \pi_1)\}.$$

We have

$$\min_{x_1, \pi_1} V(1, x_1, \pi_1) > c_{\mathrm{s}} + c_{\mathrm{o}}\delta + c_{\mathrm{in}} - K.$$

And it is clear that $U_{\mathrm{b}}(0, x_0, \pi_0) > c_{\mathrm{s}} + c_{\mathrm{o}}\delta + c_{\mathrm{in}} - K$. Since $F(\tau, x_0, \pi_0)$ decreases with π_0, if there exists a threshold $\overline{\Theta} = \max\{\overline{\Theta}_1, \overline{\Theta}_2\}$ such that

$$F(\tau, x_0, \overline{\Theta}_1) = (c_{\mathrm{s}} + c_{\mathrm{o}}\delta + c_{\mathrm{in}})/c_{\mathrm{f}},$$
$$\text{and } \Box F(\tau, x_0, \overline{\Theta}_2) = (c_{\mathrm{d}} + K)/c_{\mathrm{f}},$$

then we have $\forall \pi_0 > \overline{\Theta}$,

$$\begin{aligned} c_{\mathrm{u}}(x_0, \pi_0) &= -K + c_{\mathrm{f}} F(\tau, x_0, \pi_0) \\ &< \min\{-K + c_{\mathrm{s}} + c_{\mathrm{o}}\delta + c_{\mathrm{in}}, c_{\mathrm{d}}\} \\ &= \min\{U_{\mathrm{b}}(0, x_0, \pi_0), c_{\mathrm{d}}\}. \end{aligned}$$

This finishes the proof.

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