# Nonmonotonic propositional logic

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# Monotonic Gentzen deduction system

A sequent $\Gamma \Rightarrow \Delta$ is valid, denoted by $\models_{\mathbf{G}^1} \Gamma \Rightarrow \Delta$, if for any assignment $v$, $v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1$.

There is a monotonic, sound and complete Gentzen deduction system $\mathbf{G}^{\mathrm{a}}$ for sequents.

# Monotonic Gentzen deduction system

Gentzen deduction system $\mathbf{G}^{\mathrm{a}}$ consists of the following axioms and deduction rules:

- Axioms:

$$(\mathbf{A}_{\Rightarrow})\ \frac{\Gamma \cap \Delta \neq \emptyset \text{ or } \mathrm{incon}(\Gamma) \text{ or } \mathrm{incon}(\Delta)}{\Gamma \Rightarrow \Delta},$$

where $\Delta, \Gamma$ are sets of literals.

- Deduction rules:

$$(\Rightarrow\neg\neg^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, \neg\neg A_1 \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow \neg\neg B_1, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_2^L)\ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\vee_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\neg\wedge^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta \quad \Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \wedge A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\wedge_1^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\wedge_2^R)\ \frac{\Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\vee_1^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta \quad \Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

$$(\Rightarrow\neg\vee_2^L)\ \frac{\Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta}$$

# Nonmonotonic Gentzen deduction system

A co-sequent $\Gamma \mapsto \Delta$ is valid, denoted by $\models_{\mathbf{G_1}} \Gamma \mapsto \Delta$, if there is an assignment $v$ such that $v \models \Gamma$ and $v \not\models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \not\models \Delta$ if for every $B \in \Delta, v(B) = 0$.

There is a nonmonotonic, sound and complete Gentzen deduction system $\mathbf{G_a}$ for co-sequents.

# Nonmonotonic Gentzen deduction system

Gentzen deduction system $\mathbf{G}_{\rm a}$ consists of the following axioms and deduction rules:

- Axioms:

$$(\mathbf{A}_{\mapsto})\ \frac{\Gamma \cap \Delta = \emptyset \& \mathrm{con}(\Gamma) \& \mathrm{con}(\Delta)}{\Gamma \mapsto \Delta},$$

where $\Delta, \Gamma$ are sets of literals.

- Deduction rules:

$$(_{\mapsto}\neg\neg^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, \neg\neg A_1 \mapsto \Delta} \qquad (_{\mapsto}\neg\neg^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto \neg\neg B_1, \Delta}$$

$$(_{\mapsto}\wedge^L)\ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (_{\mapsto}\wedge^R_1)\ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_{\mapsto}\wedge^R_2)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_{\mapsto}\vee^L_1)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (_{\mapsto}\vee^R)\ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(_{\mapsto}\vee^L_2)\ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta}$$

$$(_{\mapsto}\neg\wedge^L_1)\ \frac{\Gamma, \neg A_1 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (_{\mapsto}\neg\wedge^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta \quad \Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(_{\mapsto}\neg\wedge^L_2)\ \frac{\Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta}$$

$$(_{\mapsto}\neg\vee^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta \quad \Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \vee A_2) \mapsto \Delta} \qquad (_{\mapsto}\neg\vee^R_1)\ \frac{\Gamma \mapsto \neg B_1, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

$$(_{\mapsto}\neg\vee^R_2)\ \frac{\Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

# Other definitions: sequents

A sequent $\Gamma \Rightarrow \Delta$ is valid if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where

$$v \models \Gamma \text{ if for every } A \in \Gamma, v(A) = 1\text{; and } v \models \Delta \text{ if } \begin{cases} \text{for every } B \in \Delta, v(B) = 1 \\ \text{for some } B \in \Delta, v(B) = 0 \\ \text{for every } B \in \Delta, v(B) = 0; \end{cases}$$

There are sound and complete Gentzen deduction systems $\mathbf{G}^{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}$ for sequents.

| definition | system |
|---|---|
| $\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \forall B \in \Delta(v(B) = 1))$ | $\mathbf{G}^{\mathrm{b}}$ |
| $\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \exists B \in \Delta(v(B) = 0))$ | $\mathbf{G}^{\mathrm{c}}$ |
| $\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \forall B \in \Delta(v(B) = 0))$ | $\mathbf{G}^{\mathrm{d}}$; |

# Other definitions: co-sequents

- A co-sequent $\Gamma \mapsto \Delta$ is valid if there is an assignment $v$ such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if $\begin{cases} \text{for some } B \in \Delta, v(B) = 0 \\ \text{for every } B \in \Delta, v(B) = 1 \\ \text{for some } B \in \Delta, v(B) = 1. \end{cases}$

There are sound and complete Gentzen deduction systems $\mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}_{\mathrm{d}}$ for co-sequents.

| definition | system |
|---|---|
| $\exists v(\forall A \in \Gamma(v(A) = 1) \& \exists B \in \Delta(v(B) = 0))$ | $\mathbf{G}_{\mathrm{b}}$ |
| $\exists v(\forall A \in \Gamma(v(A) = 1) \& \forall B \in \Delta(v(B) = 1))$ | $\mathbf{G}_{\mathrm{c}}$ |
| $\exists v(\forall A \in \Gamma(v(A) = 1) \& \exists B \in \Delta(v(B) = 1))$ | $\mathbf{G}_{\mathrm{d}}$. |

# Monotonicity

(1) $\mathbf{G}^{\mathrm{a}}, \mathbf{G}^{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}$ are monotonic in $\Gamma$;

(2) $\mathbf{G}_{\mathrm{a}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}_{\mathrm{d}}$ are nonmonotonic in $\Gamma$;

(3) $\mathbf{G}_{\mathrm{a}}, \mathbf{G}^{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}$ are monotonic in $\Delta$;

(4) $\mathbf{G}^{\mathrm{a}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}_{\mathrm{d}}$ are nonmonotonic in $\Delta$.

# Axioms

| system | precondition | mono in $\Gamma$ | mono in $\Delta$ |
|---|---|---|---|
| $\mathbf{G}^{\mathrm{a}}$ | $\Gamma \cap \Delta \neq \emptyset$ or $\mathrm{incon}(\Gamma)$ or $\mathrm{incon}(\Delta)$ | $Y$ | $Y$ |
| $\mathbf{G}_{\mathrm{a}}$ | $\Gamma \cap \Delta = \emptyset \& \mathrm{con}(\Gamma) \& \mathrm{con}(\Delta)$ | $N$ | $N$ |
| $\mathbf{G}^{\mathrm{b}}$ | $\Delta \subseteq \Gamma$ or $\mathrm{incon}(\Gamma)$ | $Y$ | $N$ |
| $\mathbf{G}_{\mathrm{b}}$ | $\Delta \not\subseteq \Gamma \& \mathrm{con}(\Gamma)$ | $N$ | $Y$ |
| $\mathbf{G}^{\mathrm{c}}$ | $\Gamma \cap \neg\Delta \neq \emptyset$ or $\mathrm{incon}(\Gamma)$ or $\mathrm{incon}(\neg\Delta)$ | $Y$ | $Y$ |
| $\mathbf{G}_{\mathrm{c}}$ | $\Gamma \cap \neg\Delta = \emptyset \& \mathrm{con}(\Gamma) \& \mathrm{con}(\Delta)$ | $N$ | $N$ |
| $\mathbf{G}^{\mathrm{d}}$ | $\neg\Delta \subseteq \Gamma$ or $\mathrm{incon}(\Gamma)$ | $Y$ | $N$ |
| $\mathbf{G}_{\mathrm{d}}$ | $\neg\Delta \not\subseteq \Gamma \& \mathrm{con}(\Gamma)$ | $N$ | $Y$ |

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