An FPRAS for the four-vertex model without ARS on planar graphs

Junda Li, Zhiguo Fu

Frontiers of Computer Science, DOI: 10.1007/s11704-026-51888-1

Problems & Ideas

- Problem : Designing an FPRAS[1] for the planar four-vertex model without ARS[2]

  For the four-vertex model without ARS on planar graphs, it is difficult to analyze the mixing speed of the Markov chains on the set of vertices via classical methods such as canonical paths.

- Ideas:

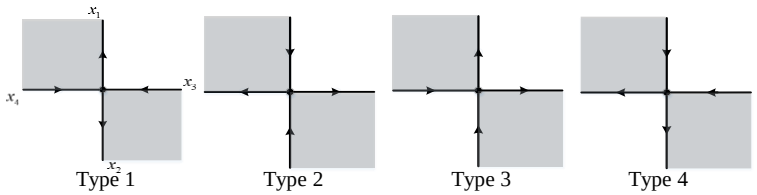
  By the circuit decomposition, we transform the four-vertex model into the hard-core model and design a Markov chain defined on the set of circuits. This is the key to overcoming the ARS limitation for the four-vertex model on planar graphs.

Circuit decomposition

We decompose the underlying graph into several circuits and replace the assignment on each edge of a black face with an assignment on its corresponding directed circuit.

Decomposition method: An arbitrary 4-regular planar graph G can be 2-colored. The boundary of each black face forms a circuit.

Assignment replacement: Based on the above facts, the local orientations can be categorized into four types, as shown in the following figure. We define the set of all circuits as $\mathcal{C}$.



Type 1 Type 2 Type 3 Type 4

Remark: These four types are in one-to-one correspondence with the four local types in the four-vertex model.

Markov chain $\{Z_t\}$ on the state space $\Omega \subseteq \{0,1\}^{\mathcal{C}}$

Based on the new partition function, the transition probability matrix P of $\{Z_t\}$ is as follows:

- Choose an arbitrary circuit $C_i \in \mathcal{C}$ at random.
- If $\sigma(\Gamma(C_i)) = \mathbf{0}$, where $\mathbf{0}$ denotes the all-zero string of length δ_{C_i}, then:

$$\sigma' = \begin{cases} \text{only take } C_i \text{ to 1 in } \sigma \text{ with the probability } \frac{\delta_{c_i}}{1+\delta_{c_i}}; \\ \text{only take } C_i \text{ to 0 in } \sigma \text{ with the probability } \frac{1}{1+\delta_{c_i}}. \end{cases}$$

- else: set $\sigma' \leftarrow \sigma$.
- The new state is σ'.

Remark: $\Gamma(C_i)$ is the set of neighbors of C_i and $\delta_{C_i} = |\Gamma(C_i)|$.

1: FPRAS, fully polynomial randomized approximation scheme.

2: ARS, arrow reversal symmetry, i.e., the weight of configurations remains unchanged if all arrows are flipped.

Main Contributions

- Contributions:
  - A connection between the four-vertex model without ARS on planar graphs and the hard-core model;
  - A Markov chain on the set of circuits for the four-vertex model on planar graphs;
  - An FPRAS is presented for the four-vertex model without ARS on three classes of planar graphs. The main theorems related to FPRAS are as follows:

Theorem 1. For the four-vertex model with the con-straint function f^*, whose underlying gr-aph is two-by-two-intersection free under the circuit decomposition $\mathcal{C}$, the Markov chain $\{Z_t\}$ mixes in time:

$$O\left(\frac{n(1+b^{\delta})}{b^{\delta}+1-b\delta}\log\left(\frac{n\delta}{\varepsilon}\right)\right),$$

for $\delta \geq 2,\ b \leq \frac{1}{\delta}$.

Theorem 2. For the four-vertex model with the constraint function f^*, whose underlying graph is $S_{1,1,t}$-free ($t \geq 2$) under the circuit decomposition $\mathcal{C}$,

- If $\delta \geq 3$ and $0 < b \leq 1$, the Markov chain $\{Z_t\}$ mixes in time:

$$\left(\frac{\delta\left(1+b^{\delta}\right)^{\delta}(1+b)}{b^{\delta}}\right)^{O\left(\frac{2\left(1+b^{\delta}\right)^{2\delta}(1+b)^{2\delta+2}}{b^{2\delta}}+1\right)} \times O\left(n\log\left(\frac{n}{\varepsilon}\right)\right).$$

- If $\delta \geq 3$ and $b > 1$, the Markov chain $\{Z_t\}$ mixes in time:

$$\left(\frac{\delta(1+b)\left(1+b^{\delta}\right)^{\delta}}{b}\right)^{O\left(\frac{2(1+b)^{2}\left(1+b^{\delta}\right)^{4\delta}}{b^{2\delta}}+1\right)} \times O\left(n\log\left(\frac{n}{\varepsilon}\right)\right).$$

Theorem 3. For the four-vertex model with the constraint function f^*, whose underlying graph is composed of blow-ups of paths and cycles under the circuit decomposition $\mathcal{C}$,

- If $\delta \geq 3$ and $0 < b \leq 1$, the Markov chain $\{Z_t\}$ mixes in time:

$$\left(\frac{\delta\left(1+b^{\delta}\right)^{\delta}(1+b)}{b^{\delta}}\right)^{O\left(\frac{2(\delta-1)\left(1+b^{\delta}\right)^{2\delta}(1+b)^{2\delta(\delta-1)+2}}{b^{2\delta}}+1\right)} \times O\left(n\log\left(\frac{n}{\varepsilon}\right)\right).$$

- If $\delta \geq 3$ and $b > 1$, the Markov chain $\{Z_t\}$ mixes in time:

$$\left(\frac{\delta(1+b)\left(1+b^{\delta}\right)^{\delta}}{b}\right)^{O\left(\frac{2(\delta-1)(1+b)^{2}\left(1+b^{\delta}\right)^{2\delta^{2}}}{b^{2\delta}}+1\right)} \times O\left(n\log\left(\frac{n}{\varepsilon}\right)\right).$$