Fault-Tolerant Encoding and Decoding Method in Steane Code: Comprehensive Analysis of Maximum Threshold

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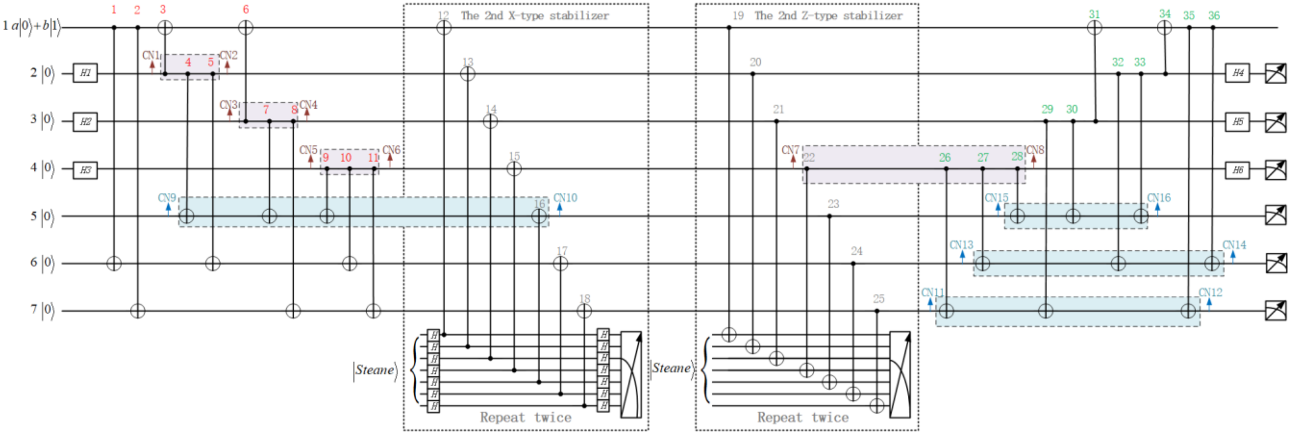
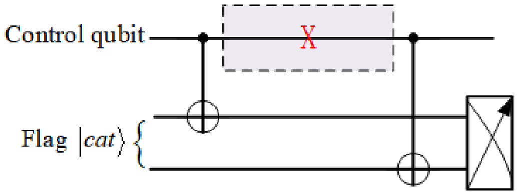
Problems & Ideas

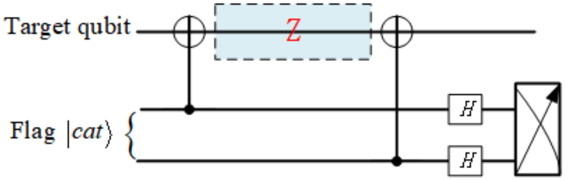
- Problems of error propagation in the original Steane code:
  - Prior work generally assumes that these encoding/decoding processes are perfect, leading to an incomplete threshold analysis.
  - There are related works that apply the flag-based syndrome method to encoding, but they assume that everything except coding is perfect, that is, they do not consider the errors introduced by the flag-based syndrome method and auxiliary quantum gates.
- Ideas: We systematically introduce fault-tolerant mechanisms in each stage of the fault-tolerant Steane code for the first time.
  - We improve the flag-based syndrome method to prevent it from introducing additional error propagation itself and integrate this improvement into our work.
  - We propose a fault-tolerant encoding and decoding method in Steane code, considering errors on each quantum operation.



Fault-tolerant encoding and decoding method



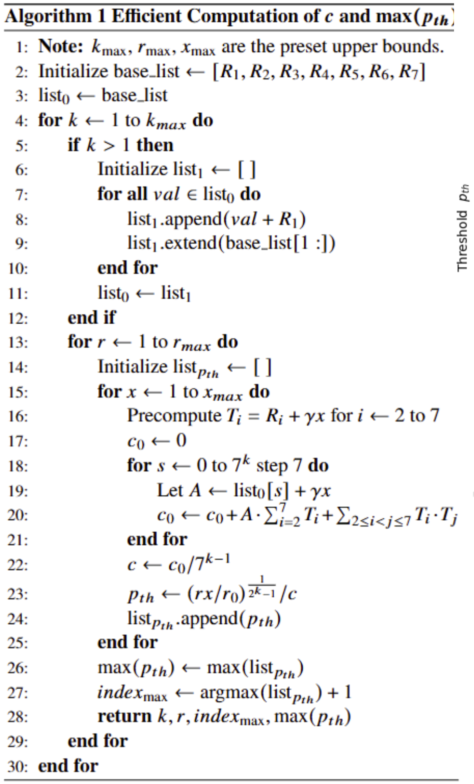
(a) Flag to detect the X error on the control qubit



(b) Flag to detect the Z error on the target qubit

Main Contributions

- Contributions:
  - Combined with the results of measuring stabilizers and redundant qubits after decoding, most error cases can be identified within an error correction period.
  - Different errors may lead to the same measurement results due to error propagation. To address this, we apply the improved flag-based syndrome method, aiming to minimize the interference.
  - We propose an algorithm based on logical depth to estimate the threshold and obtain conclusions through simulation.



Algorithm 1 **Efficient Computation of c and $\max(p_{th})$**

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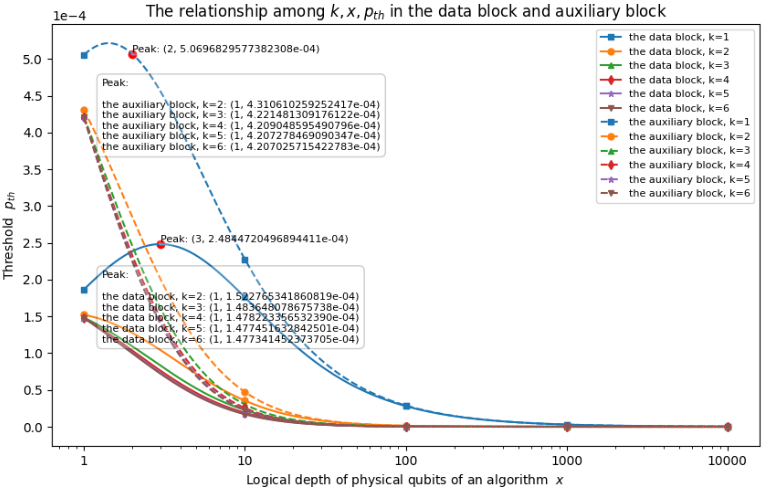
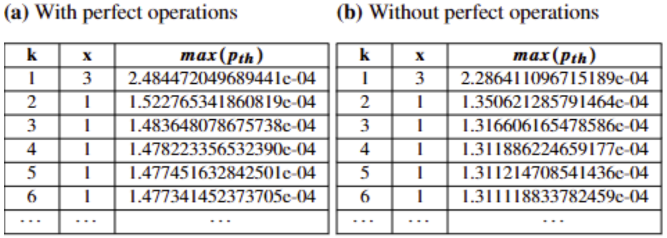
Note: $k_{\max}, r_{\max}, x_{\max}$ are the preset upper bounds.
Initialize base_list $\leftarrow [R_1, R_2, R_3, R_4, R_5, R_6, R_7]$
$\text{list}_0 \leftarrow$ base_list
for $k \leftarrow 1$ to $k_{max}$ do
    if $k > 1$ then
        Initialize $\text{list}_1 \leftarrow [\,]$
        for all $val \in \text{list}_0$ do
            $\text{list}_1.\text{append}(val + R_1)$
            $\text{list}_1.\text{extend}(\text{base\_list}[1:])$
        end for
        $\text{list}_0 \leftarrow \text{list}_1$
    end if
    for $r \leftarrow 1$ to $r_{max}$ do
        Initialize $\text{list}_{p_{th}} \leftarrow [\,]$
        for $x \leftarrow 1$ to $x_{max}$ do
            Precompute $T_i = R_i + \gamma x$ for $i \leftarrow 2$ to 7
            $c_0 \leftarrow 0$
            for $s \leftarrow 0$ to $7^k$ step 7 do
                Let $A \leftarrow \text{list}_0[s] + \gamma x$
                $c_0 \leftarrow c_0 + A \cdot \sum_{i=2}^{7} T_i + \sum_{2 \le i < j \le 7} T_i \cdot T_j$
            end for
            $c \leftarrow c_0 / 7^{k-1}$
            $p_{th} \leftarrow (rx/r_0)^{\frac{1}{2^k-1}} / c$
            $\text{list}_{p_{th}}.\text{append}(p_{th})$
        end for
        $\max(p_{th}) \leftarrow \max(\text{list}_{p_{th}})$
        $index_{\max} \leftarrow \text{argmax}(\text{list}_{p_{th}}) + 1$
        return $k, r, index_{\max}, \max(p_{th})$
    end for
end for
```

Algorithm based on quantum logical depth.



The relationship among k,x,p_{th} in the data block and auxiliary block.



(a) With perfect operations

k	**x**	$\boldsymbol{max(p_{th})}$
1	3	2.484472049689441e-04
2	1	1.522765341860819e-04
3	1	1.483648078675738e-04
4	1	1.478223356532390e-04
5	1	1.477451632842501e-04
6	1	1.477341452373705e-04
⋯	⋯	⋯

(b) Without perfect operations

k	**x**	$\boldsymbol{max(p_{th})}$
1	3	2.286411096715189e-04
2	1	1.350621285791464e-04
3	1	1.316606165478586e-04
4	1	1.311886224659177e-04
5	1	1.311214708541436e-04
6	1	1.311118833782459e-04
⋯	⋯	⋯

The relationship among k,x,max(p_{th}) in the data block.

The figure and the table show only up to six levels of concatenated Steane code.

Conclusion: Our simulation results show that the logical depth of a physical qubit for an error correction period is to only execute one algorithmic fault-tolerant quantum operation when using the concatenated Steane code (k>1). The error between the circuit performance of the error model with partial perfect operation assumptions and that without perfect operation assumption is very small. Our error model does not affect the $O(10^{-4})$ order of the maximum threshold and the logical depth of physical qubits in the executable algorithm corresponding to the maximum threshold. This shows the performance of the fault-tolerant Steane code as much as possible while taking error propagation into account.