

Monotonicity and Nonmonotonicity In L_3 -Valued Propositional Logic

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Frontiers of Computer Science, DOI: 10.1007/s11704-021-0382-0

Three-valued logic

Three-valued logic should be different from two-valued logic [1,2].

In two-valued logic [6], a sequent $\Gamma \Rightarrow \Delta$ is valid if for any assignment v , either

for some formula $A \in \Gamma, v(A) = \mathbf{f}$,

or

for some formula $B \in \Delta, v(B) = \mathbf{t}$.

The negation of validity is the validity of co-sequent $\Gamma \mapsto \Delta$, that is, there is an assignment v such that

for each formula $A \in \Gamma, v(A) \neq \mathbf{f}$,

and

for each formula $B \in \Delta, v(B) \neq \mathbf{t}$.

Multisequents

A multisequent (many-placed sequent) $\Delta|\Theta|\Gamma$ is $\mathbf{G}^{\exists\exists\exists}$ -valid if for any assignment v , one of the following conditions holds:

$$\left\{ \begin{array}{l} \text{for some formula } A \in \Delta, v(A) = \mathbf{t}; \\ \text{for some formula } B \in \Theta, v(B) = \mathbf{m}; \\ \text{for some formula } C \in \Delta, v(C) = \mathbf{f}. \end{array} \right.$$

The negation of validity is the validity of co-multisequent $\Delta : \Theta : \Gamma$, which is $\mathbf{G}_{\forall\forall\forall}$ -valid if there is an assignment v such that

$$\left[\begin{array}{l} \text{for each formula } A \in \Delta, v(A) \neq \mathbf{t}; \\ \text{for each formula } B \in \Theta, v(B) \neq \mathbf{m}; \\ \text{for each formula } C \in \Delta, v(C) \neq \mathbf{f}, \end{array} \right.$$

where $\left[\begin{array}{l} \delta_1 \\ \delta_2 \\ \delta_3 \end{array} \right.$ means that δ_1, δ_2 and δ_3 hold; and $\left\{ \begin{array}{l} \delta_1 \\ \delta_2 \\ \delta_3 \end{array} \right.$ means that either δ_1 or δ_2 or δ_3 holds.

A general case

A multisequent $\Delta|\Theta|\Gamma$ is $\mathbf{G}^{Q_1Q_2Q_3}$ -valid, denoted by $\models^{Q_1Q_2Q_3} \Delta|\Theta|\Gamma$, if for any assignment v ,

$$\left\{ \begin{array}{l} Q_1A \in \Delta(v(A) = \mathbf{t}) \\ Q_2B \in \Theta(v(B) = \mathbf{m}) \\ Q_3C \in \Delta(v(C) = \mathbf{f}); \end{array} \right.$$

and a co-multisequent $\Delta : \Theta : \Gamma$ is $\mathbf{G}_{Q_1Q_2Q_3}$ -valid, denoted by $\models_{Q_1Q_2Q_3} \Delta|\Theta|\Gamma$, if there is an assignment v such that

$$\left[\begin{array}{l} Q_1A \in \Delta(v(A) \neq \mathbf{t}) \\ Q_2B \in \Theta(v(B) \neq \mathbf{m}) \\ Q_3C \in \Delta(v(C) \neq \mathbf{f}), \end{array} \right.$$

where $Q_1, Q_2, Q_3 \in \{\forall, \exists\}$.

$\models^{Q_1Q_2Q_3} \Delta|\Theta|\Gamma$ is complementary to $\models_{Q'_1Q'_2Q'_3} \Delta : \Theta : \Gamma$, that is, for any sets Δ, Θ, Γ of formulas, either $\models^{Q_1Q_2Q_3} \Delta|\Theta|\Gamma$ or $\models_{Q'_1Q'_2Q'_3} \Delta : \Theta : \Gamma$, where $\forall' = \exists, \exists' = \forall$.

A general case

A general case:

There are 8 validities of multisequents and 8 validities of co-multisequents [6,7], and there are corresponding Gentzen deduction systems:

system	Q_1	Q_2	Q_3	co-system	Q_1	Q_2	Q_3
G^{EEE}	\exists	\exists	\exists	$G^{\forall\forall\forall}$	\forall	\forall	\forall
G^{AEE}	\exists	\exists	\forall	$G^{\forall\forall\exists}$	\forall	\forall	\exists
G^{EAE}	\exists	\forall	\exists	$G^{\forall\exists\forall}$	\forall	\exists	\forall
G^{EEA}	\forall	\exists	\exists	$G^{\forall\exists\forall}$	\exists	\forall	\forall
G^{AAE}	\exists	\forall	\forall	$G^{\exists\forall\exists}$	\forall	\exists	\exists
G^{AEA}	\forall	\exists	\forall	$G^{\exists\forall\exists}$	\exists	\forall	\exists
G^{EAA}	\forall	\forall	\exists	$G^{\exists\exists\forall}$	\exists	\exists	\forall
$G^{\forall\forall\forall}$	\forall	\forall	\forall	$G^{\exists\exists\exists}$	\exists	\exists	\exists

Gentzen deduction systems

Gentzen deduction systems are different for 16 cases.

Deduction rules for logical connectives for $\mathbf{G}^{Q_1 Q_2 Q_3}$ are same; and the deduction rules for logical connectives for $\mathbf{G}_{Q_1 Q_2 Q_3}$ are same. The only differences are in axioms.

Axioms 1

$$\begin{aligned}
 (\mathbf{A}_{\exists\exists\exists}) & \frac{\Delta' \cap \Theta' \cap \Gamma' \neq \emptyset}{\Delta | \Theta | \Gamma} \\
 (\mathbf{A}_{\forall\forall\forall}) & \frac{\Delta' \cap \Theta' \cap \Gamma' = \emptyset}{\Delta : \Theta : \Gamma},
 \end{aligned}$$

where $\Delta | \Theta | \Gamma$ is literal, $\sigma(\Delta' | \Theta' | \Gamma') = \Delta' | \Theta' | \Gamma'$ is atomic, and σ consists of the following rules:

$$\begin{array}{ll}
 (\sim^L) \frac{\Delta | p, \Theta | \Gamma}{\Delta, \sim p | \Theta | \Gamma} & (\sim\sim^L) \frac{\Delta | \Theta | p, \Gamma}{\Delta, \sim\sim p | \Theta | \Gamma} \\
 (\sim^M) \frac{\Delta | \Theta | p, \Gamma}{\Delta | \sim p, \Theta | \Gamma} & (\sim\sim^M) \frac{\Delta, p | \Theta | \Gamma}{\Delta | \sim\sim p, \Theta | \Gamma} \\
 (\sim^R) \frac{\Delta, p | \Theta | \Gamma}{\Delta | \Theta | \sim p, \Gamma} & (\sim\sim^R) \frac{\Delta | p, \Theta | \Gamma}{\Delta | \Theta | \sim\sim p, \Gamma}
 \end{array}$$

Axioms 2

$$\begin{aligned} (\mathbf{A}_{\exists\forall}) & \frac{\Delta \text{ or } \Theta=\emptyset \text{ or } \Gamma=\emptyset \text{ or } (\Gamma\cap\Theta\subseteq\Delta \& (\sim\sim\Theta\subseteq\Delta \text{ or } \sim\Gamma\subseteq\Delta))}{\Delta|\Theta|\Gamma} \\ (\mathbf{A}_{\forall\exists}) & \frac{\Delta\&\Theta\neq\emptyset\neq\Gamma\&(\Gamma\cap\Theta\subseteq\Delta \text{ or } \sim\sim\Theta\subseteq\Delta\&\sim\Gamma\subseteq\Delta)}{\Delta:\Theta:\Gamma} \end{aligned}$$

where $\Delta|\Theta|\Gamma$ is literal.

Axioms 3

$$\begin{aligned} (\mathbf{A}_{\exists\forall}) & \frac{\Delta \text{ or } \theta=\emptyset \text{ or } \Gamma=\emptyset \text{ or } (\Gamma\cap\theta\subseteq\Delta \& (\sim\sim\theta\subseteq\Delta \text{ or } \sim\Gamma\subseteq\Delta))}{\Delta|\theta|\Gamma} \\ (\mathbf{A}_{\forall\exists}) & \frac{\Delta\&\theta\neq\emptyset\neq\Gamma\&(\Gamma\cap\theta\subseteq\Delta \text{ or } \sim\sim\theta\subseteq\Delta\&\sim\Gamma\subseteq\Delta)}{\Delta:\theta:\Gamma} \end{aligned}$$

where $\Delta|\theta|\Gamma$ is literal.

Axioms 4

$$\begin{aligned} (\mathbf{A}^{\forall\forall\forall}) & \frac{\Delta'=\emptyset \text{ or } \Theta'=\emptyset \text{ or } \Gamma'=\emptyset \text{ or } \Delta'=\Theta'=\Gamma'=\{p\}}{\Delta|\Theta|\Gamma} \\ (\mathbf{A}^{\exists\exists\exists}) & \frac{\Delta'\neq\emptyset\neq\Theta'\&\Gamma'\neq\emptyset\&(\Delta'\neq\Theta' \text{ or } \Delta'\neq\Gamma' \text{ or } \Theta'\neq\Gamma')}{\Delta:\Theta:\Gamma} \end{aligned}$$

where $\Delta|\Theta|\Gamma$ is literal, and $\sigma(\Delta'|\Theta'|\Gamma') = \Delta'|\Theta'|\Gamma'$ is atomic.

Gentzen deduction systems

system	mono in Δ	mono in Θ	mono in Γ
$\mathbf{G}^{\exists\exists\exists} = \mathbf{A}^{\exists\exists\exists} + \mathbf{G}^{\exists\exists\exists}$	<i>Y</i>	<i>Y</i>	<i>Y</i>
$\mathbf{G}^{\exists\exists\forall} = \mathbf{A}^{\exists\exists\forall} + \mathbf{G}^{\exists\exists\forall}$	<i>Y</i>	<i>Y</i>	<i>N</i>
$\mathbf{G}^{\exists\forall\forall} = \mathbf{A}^{\exists\forall\forall} + \mathbf{G}^{\exists\forall\forall}$	<i>Y</i>	<i>N</i>	<i>N</i>
$\mathbf{G}^{\forall\forall\forall} = \mathbf{A}^{\forall\forall\forall} + \mathbf{G}^{\forall\forall\forall}$	<i>N</i>	<i>N</i>	<i>N</i>

Gentzen deduction systems

co-system	mono in Δ	mono in Θ	mono in Γ
$\mathbf{G}_{\forall\forall\forall} = \mathbf{A}_{\forall\forall\forall} + \mathbf{G}^{\neq}$	<i>N</i>	<i>N</i>	<i>N</i>
$\mathbf{G}_{\forall\forall\exists} = \mathbf{A}_{\forall\forall\exists} + \mathbf{G}^{\neq}$	<i>N</i>	<i>N</i>	<i>Y</i>
$\mathbf{G}_{\forall\exists\exists} = \mathbf{A}_{\forall\exists\exists} + \mathbf{G}^{\neq}$	<i>N</i>	<i>Y</i>	<i>Y</i>
$\mathbf{G}_{\exists\exists\exists} = \mathbf{A}_{\exists\exists\exists} + \mathbf{G}^{\neq}$	<i>Y</i>	<i>Y</i>	<i>Y</i>

References

- [1] Avron A. Natural 3-valued logics: Characterization and proof theory [J]. J. of Symbolic Logic, 1991, 56:276-294.
- [2] Avron, A., Gentzen-type systems, resolution and tableaux [J]. Journal of Automated Reasoning, 1993, 10:265-281.
- [3] Baaz M. Fermüller C. G. Salzer G. and Zach R. Labeled Calculi and Finite-Valued Logics [J]. Studia Logica, 1998, 61:7-33.
- [4] Fitting M. C. Many-valued modal logics, I,II [J]. Fundamenta Informaticae, 1991, 15:235-254; 1992, 17:55-73.
- [5] Li W. Mathematical Logic, Foundations for Information Science [M]. Progress in Computer Science and Applied Logic, vol.25, Birkhäuser, 2010.
- [6] Li W. and Sui Y. Multisequent Gentzen deduction systems for B_2^2 -valued first-order logic [J]. Artificial Intelligence Research, 2018, 7:53-.
- [7] Malinowski G. Many-valued Logic and its Philosophy [M]. D. M. Gabbay and J. Woods (eds.) Handbook of the History of Logic, vol. 8, The Many Valued and Nonmonotonic Turn in Logic, Elsevier, 2009.
- [8] Mateus P. Rasga J. and Sernadas C. Modal sequent calculi labelled with truth values: cut elimination [J]. Logic Journal of the IGPL, 2005, 13:173-199.
- [9] Post E. L. Determination of all closed systems of truth tables [J]. Bulletin American Mathematical Society, 1920, 26:437.
- [10] Urquhart A. Basic many-valued logic [M]. D. Gabbay, F. Guentner (eds.) Handbook of Philosophical Logic, vol. 2 (2d edition), Dordrecht: Kluwer, 249-295, 2001.
- [11] Wansing H. The power of Belnap: sequent systems for SIXTEEN3 [J]. J. Philosophical Logic, 2010, 39:369-393.
- [12] Zach R. Proof theory of finite-valued logics [R]. Technical Report TUW-E185.2-Z.1-93, Institut Für Computersprachen, Technische Universität Wien, 1993.