

Monotonic and nonmonotonic gentzen
deduction systems for L_3 -valued
propositional logic

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Multisequents and co-multisequents

Given sets Δ, Θ, Γ of formulas, a multisequent $\Delta \mid \Theta \mid \Gamma$ is valid if for any assignment v , either $v \models \Delta$, or $v \models \Theta$, or $v \models \Gamma$, where

$$\begin{aligned}v \models \Delta &\text{ iff } \mathbf{E}A \in \Delta (v(A) = \mathbf{t}), \\v \models \Theta &\text{ iff } \mathbf{E}B \in \Theta (v(B) = \mathbf{m}), \\v \models \Gamma &\text{ iff } \mathbf{E}C \in \Gamma (v(C) = \mathbf{f}).\end{aligned}$$

A co-multisequent $\Delta : \Theta : \Gamma$ is satisfiable if there is an assignment v such that $v \not\models \Delta$, $v \not\models \Theta$, and $v \not\models \Gamma$, where

$$\begin{aligned}v \not\models \Delta &\text{ iff } \mathbf{A}A \in \Delta (v(A) \neq \mathbf{t}), \\v \not\models \Theta &\text{ iff } \mathbf{A}B \in \Theta (v(B) \neq \mathbf{m}), \\v \not\models \Gamma &\text{ iff } \mathbf{A}C \in \Gamma (v(C) \neq \mathbf{f}).\end{aligned}$$

Monotonic and nonmonotonic systems

Monotonic system \mathbf{G} : the axiom is of form

$$\frac{\Delta' \cap \Theta' \cap \Gamma' \neq \emptyset}{\Delta \mid \Theta \mid \Gamma},$$

where Δ, Θ, Γ are sets of literals; $\Delta', \Theta', \Gamma'$ are sets of variables and $\Delta' \mid \Theta' \mid \Gamma'$ is a variant form of $\Delta \mid \Theta \mid \Gamma$.

Nonmonotonic system \mathbf{G}^- : the axiom is of form

$$\frac{\Delta' \cap \Theta' \cap \Gamma' = \emptyset}{\Delta : \Theta : \Gamma}.$$

Soundness and completeness theorem

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- (1) $\Delta \mid \Theta \mid \Gamma$ is provable in \mathbf{G} if and only if $\Delta \mid \Theta \mid \Gamma$ is valid;
 - (2) $\Delta : \Theta : \Gamma$ is provable in \mathbf{G}^- if and only if $\Delta : \Theta : \Gamma$ is satisfiable.

Cuts

Let $*_1, *_2 \in \{\lambda, \sim, \sim\sim\}$ with $*_1 \neq *_2$.

(1) The first class of cuts:

$$(C_{11}) \frac{\Delta_1, *_1 A | \Theta_1 | \Gamma_1 \quad \Delta_2, *_2 A | \Theta_2 | \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

$$(C_{12}) \frac{\Delta_1, *_1 A | \Theta_1 | \Gamma_1 \quad \Delta_2 | \Theta_2, \sim *_2 A | \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

$$(C_{13}) \frac{\Delta_1, *_1 A | \Theta_1 | \Gamma_1 \quad \Delta_2 | \Theta_2 | \sim\sim *_2 A, \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

Cuts

(2) The second class of cuts:

$$(C_{21}) \frac{\Delta_1 | \Theta_1, *_1 A | \Gamma_1 \quad \Delta_2, \sim \sim *_2 A | \Theta_2 | \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

$$(C_{22}) \frac{\Delta_1 | \Theta_1, *_1 A | \Gamma_1 \quad \Delta_2 | \Theta_2, *_2 A | \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

$$(C_{23}) \frac{\Delta_1 | \Theta_1, *_1 A | \Gamma_1 \quad \Delta_2 | \Theta_2 | \sim *_2 A, \Gamma_2}{\Delta_1, \Delta_2 | \Theta_1, \Theta_2 | \Gamma_1, \Gamma_2}$$

Cuts

(3) The third class of cuts:

$$\begin{aligned} (\mathbf{C}_{31}) \quad & \frac{\Delta_1|\Theta_1| *_{1} A, \Gamma_1 \quad \Delta_2, \sim *_{2} A|\Theta_2|\Gamma_2}{\Delta_1, \Delta_2|\Theta_1, \Theta_2|\Gamma_1, \Gamma_2} \\ (\mathbf{C}_{32}) \quad & \frac{\Delta_1|\Theta_1| *_{1} A, \Gamma_1 \quad \Delta_2|\Theta_2, \sim\sim *_{2} A|\Gamma_2}{\Delta_1, \Delta_2|\Theta_1, \Theta_2|\Gamma_1, \Gamma_2} \\ (\mathbf{C}_{33}) \quad & \frac{\Delta_1|\Theta_1| *_{1} A, \Gamma_1 \quad \Delta_2|\Theta_2| *_{2} A, \Gamma_2}{\Delta_1, \Delta_2|\Theta_1, \Theta_2|\Gamma_1, \Gamma_2}, \end{aligned}$$

Cut elimination theorem

Let \mathbf{G}^+ be \mathbf{G} plus the cuts.

Theorem(Cut elimination theorem). For any multisequent $\Delta|\Theta|\Gamma$, if $\vdash^+ \Delta|\Theta|\Gamma$ then $\vdash \Delta|\Theta|\Gamma$.

Variants

Given sets Δ, Θ, Γ of formulas, $\models \Delta! \Theta! \Gamma$ if there is an assignment v such that $v \models \Delta$, $v \models \Theta$ and $v \models \Gamma$, where

$v \models \Delta$ if for each $A \in \Delta$, $v(A) = \mathbf{t}$

$v \models \Theta$ if for each $B \in \Theta$, $v(B) = \mathbf{m}$

$v \models \Gamma$ if for each $C \in \Gamma$, $v(C) = \mathbf{f}$.

In propositional logic, $\Delta! \Gamma$ is equivalent to $\Gamma : \Delta$; and in \mathbf{L}_3 -valued propositional logic, $\Delta! \Theta! \Gamma!$ is not equivalent to $\Delta : \Theta : \Gamma$.

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