Nonmonotonic Propositional Logic[1]

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Abstract: Gentzen deduction system for sequents is monotonic in both Γ and Δ, that is, given any theories $\Gamma, \Gamma', \Delta, \Delta'$, if sequent $\Gamma \Rightarrow \Delta$ is provable and $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta'$ then $\Gamma' \Rightarrow \Delta, \Gamma \Rightarrow \Delta', \Gamma' \Rightarrow \Delta'$ are provable. The Gentzen deduction system for co-sequents is nonmonotonic. In this paper, four Gentzen deduction systems $\mathbf{G}^{\rm a}, \mathbf{G}^{\rm b}, \mathbf{G}^{\rm c}, \mathbf{G}^{\rm d}$ and their dualities $\mathbf{G}_{\rm a}, \mathbf{G}_{\rm b}, \mathbf{G}_{\rm c}, \mathbf{G}_{\rm d}$ will be given which are proved to be sound and complete with respect to definitions of the validity of sequents and co-sequents. Moreover, $\mathbf{G}^{\rm a}, \mathbf{G}^{\rm c}$ are monotonic in Γ and in Δ; and $\mathbf{G}^{\rm b}, \mathbf{G}^{\rm d}$ are monotonic in Γ and nonmonotonic in Δ. Dually, $\mathbf{G}_{\rm a}, \mathbf{G}_{\rm c}$ are nonmonotonic in Γ and monotonic in Δ; and $\mathbf{G}_{\rm b}, \mathbf{G}_{\rm d}$ are nonmonotonic in both Γ and Δ.

Keywords: Propositional logic, Gentzen deduction system, Validity, Sequent, Co-sequents.

1. Introduction

Propositional logic [1] is basic, based on which other logics are developed. The deduction system for propositional logic is monotonic, that is, given any theories Γ, Γ' and formula A, if A is deducible from Γ and Γ is a subtheory of Γ' then A is deducible from Γ' too.

Nonmonotonic logics are a class of logics which deduction systems are nonmonotonic. Typical ones are default logic [2,3], **R**-calculus [4], autoepistemic logic [5], circumscription [5], etc.

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Nonmonotonicity of a nonmonotonic logic follows from using nonmonotonic $\Delta \not\vdash A$ of a monotonic deduction $\Delta \vdash A$. We found that each nonmonotonic logic has an occurrence of $\Gamma \not\vdash A$. For example, a formula B is deducible in default logic from a default theory $(\Delta, \dfrac{A : B}{B})$ if A is deducible from Δ and $\neg B$ is not, that is,

$$\Delta \vdash A \& \Delta \not\vdash \neg B.$$

The monotonicity of $\Gamma \vdash A$ implies the nonmonotonicity of $\Delta \not\vdash A$.

As a deduction relation, $\not\vdash$ is contradictory to $\vdash$. Correspondingly, in a Gentzen deduction system, $\Gamma \Rightarrow \Delta$ is contradictory to $\Gamma \not\Rightarrow \Delta$. Therefore, as a contradictory relation $\Gamma \not\Rightarrow \Delta$, there is a nonmonotonic Gentzen deduction system $\mathbf{G}_1$ [6] such that $\mathbf{G}_1$ is sound and complete, that is, for any co-sequent $\Gamma \not\Rightarrow \Delta$, if $\Gamma \not\Rightarrow \Delta$ is provable in $\mathbf{G}_1$ then $\Gamma \not\Rightarrow \Delta$ is valid; and conversely, if $\Gamma \not\Rightarrow \Delta$ is valid then $\Gamma \not\Rightarrow \Delta$ is provable in $\mathbf{G}_1$, where $\Gamma \not\Rightarrow \Delta$ is valid if there is an assignment v such that v satisfies Γ and does not satisfy Δ.

The validity of a sequent $\Gamma \Rightarrow \Delta$ is defined as follows:

> $\models_{\mathbf{G}^1} \Gamma \Rightarrow \Delta$ if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1$.

Correspondingly, a co-sequent $\Gamma \not\Rightarrow \Delta$ being valid is defined as follows:

> $\models_{\mathbf{G}_1} \Gamma \not\Rightarrow \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \not\models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \not\models \Delta$ if for every $B \in \Delta, v(B) = 0$.

We consider other possible definitions of the validity and have the following three definitions:

• A sequent $\Gamma \Rightarrow \Delta$ is valid if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if $\begin{cases} \text{for every } B \in \Delta, v(B) = 1 \\ \text{for some } B \in \Delta, v(B) = 0 \\ \text{for every } B \in \Delta, v(B) = 0; \end{cases}$

• A co-sequent $\Gamma \mapsto \Delta$ is valid if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if $\begin{cases} \text{for some } B \in \Delta, v(B) = 0 \\ \text{for every } B \in \Delta, v(B) = 1 \\ \text{for some } B \in \Delta, v(B) = 1. \end{cases}$

Therefore, we have sound and complete Gentzen deduction systems

$$\mathbf{G}^{\mathrm{a}}(= \mathbf{G}^1), \mathbf{G}^{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}, \mathbf{G}_{\mathrm{a}}(= \mathbf{G}_1), \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}_{\mathrm{d}},$$

where

definition	system
$\forall v(\forall A \in \Gamma(v(A)=1) \Rightarrow \exists B \in \Delta(v(B)=1))$	$\mathbf{G}^{\mathrm{a}}$
$\forall v(\forall A \in \Gamma(v(A)=1) \Rightarrow \forall B \in \Delta(v(B)=1))$	$\mathbf{G}^{\mathrm{b}}$
$\forall v(\forall A \in \Gamma(v(A)=1) \Rightarrow \exists B \in \Delta(v(B)=0))$	$\mathbf{G}^{\mathrm{c}}$
$\forall v(\forall A \in \Gamma(v(A)=1) \Rightarrow \forall B \in \Delta(v(B)=0))$	$\mathbf{G}^{\mathrm{d}}$;

and

definition	system
$\exists v(\forall A \in \Gamma(v(A)=1) \& \forall B \in \Delta(v(B)=0))$	$\mathbf{G}_{\mathrm{a}}$
$\exists v(\forall A \in \Gamma(v(A)=1) \& \exists B \in \Delta(v(B)=0))$	$\mathbf{G}_{\mathrm{b}}$
$\exists v(\forall A \in \Gamma(v(A)=1) \& \forall B \in \Delta(v(B)=1))$	$\mathbf{G}_{\mathrm{c}}$
$\exists v(\forall A \in \Gamma(v(A)=1) \& \exists B \in \Delta(v(B)=1))$	$\mathbf{G}_{\mathrm{d}}$.

It will be proved that

(1) $\mathbf{G}^{\mathrm{a}}, \mathbf{G}^{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}$ are monotonic in Γ;

(2) $\mathbf{G}_{\mathrm{a}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}_{\mathrm{d}}$ are nonmonotonic in Γ;

(3) $\mathbf{G}^{\mathrm{a}}, \mathbf{G}_{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}, \mathbf{G}_{\mathrm{d}}$ are monotonic in Δ;

(4) $\mathbf{G}_{\mathrm{a}}, \mathbf{G}^{\mathrm{b}}, \mathbf{G}_{\mathrm{c}}, \mathbf{G}^{\mathrm{d}}$ are nonmonotonic in Δ.

This paper is organized as follows: the next section gives basic definitions in propositional logic; the third section gives Gentzen deduction system $\mathbf{G}^{\mathrm{a}}$ for traditional propositional logic and $\mathbf{G}_{\mathrm{a}}$ for nonmonotonic propositional logic; the next three sections give Gentzen deduction systems $\mathbf{G}^{\mathrm{b}}/\mathbf{G}_{\mathrm{b}}, \mathbf{G}^{\mathrm{c}}/\mathbf{G}_{\mathrm{c}}$ and $\mathbf{G}^{\mathrm{d}}/\mathbf{G}_{\mathrm{d}}$; and the last section concludes the whole paper with the table of monotonicity of all the systems.

Our notation is standard, and a references is [7].

2. The logical language of propositional logic

The logical language of propositional logic consists of the following symbols:
- propositional variables: $p_0, p_1, ...$;
- logical connectives: $\neg, \wedge, \vee$.

A string A of symbols is a formula if

$$A_1, A_2 ::= p|\neg A_1|A_1 \wedge A_2|A_1 \vee A_2,$$

where p is a propositional variable.

The semantics of propositional logic is given by an assignment v, a function from propositional variables to $\{0, 1\}$.

Given an assignment v, a formula A is true in v, denoted by $v \models A$, if

$$\begin{cases} v(p) = 1 & \text{if } A = p \\ v \not\models A_1 & \text{if } A = \neg A_1 \\ v \models A_1 \& v \models A_2 & \text{if } A = A_1 \wedge A_2 \\ v \models A_1 \text{ or } v \models A_2 & \text{if } A = A_1 \vee A_2, \end{cases}$$

where $\sim, \&,$ or are symbols used in meta-language, and correspondingly, $\neg, \wedge, \vee$ are the ones used in the language. Therefore, $v \not\models A_1$ can be represented as $\sim (v \models A_1)$.

Given two sets Γ, Δ of formulas, $\Gamma \Rightarrow \Delta$ is called a sequent and $\Gamma \mapsto \Delta$ is called a co-sequent.

A literal l is a propositional variable or the negation of a propositional variable. Given a set Γ of literals, Γ is inconsistent, denoted by $\text{incon}(\Gamma)$, if there is a propositional variable p such that $p, \neg p \in \Gamma$; otherwise, Γ is consistent, denoted by $\text{con}(\Gamma)$.

3. Gentzen deduction systems $\mathbf{G}^{\mathrm{a}}$ and $\mathbf{G}_{\mathrm{a}}$

A sequent $\Gamma \Rightarrow \Delta$ is $\mathbf{G}^{\mathrm{a}}$-valid, denoted by $\models_{\mathbf{G}^{\mathrm{a}}} \Gamma \Rightarrow \Delta$, if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1$.

Proposition 3.1. Let Γ and Δ be sets of literals. $\models_{\mathbf{G}^{\mathrm{a}}} \Gamma \Rightarrow \Delta$ if and only if $\Gamma \cap \Delta \neq \emptyset$ or $\text{incon}(\Gamma)$ or $\text{incon}(\Delta)$.

Proof. Assume that $\Gamma \cap \Delta \neq \emptyset$ or $\text{incon}(\Gamma)$ or $\text{incon}(\Delta)$. Then, for any assignment v, if $v \models \Gamma$ then $v \models \Delta$.

Conversely, assume that $\Gamma \cap \Delta = \emptyset \& \text{con}(\Gamma) \& \text{con}(\Delta)$. Define an assignment v such that for any propositional variable p,

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma \text{ or } \neg p \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Then, v is well-defined, and $v \not\models \Gamma \Rightarrow \Delta$.

□

Gentzen deduction system $\mathbf{G}^{\mathrm{a}}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\Rightarrow}) \ \frac{\Gamma \cap \Delta \neq \emptyset \text{ or } \text{incon}(\Gamma) \text{ or } \text{incon}(\Delta)}{\Gamma \Rightarrow \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(\Rightarrow\neg\neg^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, \neg\neg A_1 \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow \neg\neg B_1, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_2^L)\ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\vee_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\neg\wedge^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta \quad \Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \wedge A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\wedge_1^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\wedge_2^R)\ \frac{\Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\vee_1^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta \quad \Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

$$(\Rightarrow\neg\vee_2^L)\ \frac{\Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta}$$

Theorem 3.2(Soundness and completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$, $\vdash_{\mathbf{G}^{\mathrm{a}}} \Gamma \Rightarrow \Delta$ if and only if $\models_{\mathbf{G}^{\mathrm{a}}} \Gamma \Rightarrow \Delta$.

□

□

Theorem 3.3(Monotonicity theorem). $\mathbf{G}^{\mathrm{a}}$ is monotonic in both Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_{\mathrm{a}}} \Gamma \Rightarrow \Delta \text{ imply } \vdash_{\mathbf{G}_{\mathrm{a}}} \Gamma' \Rightarrow \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_{\mathrm{a}}} \Gamma \Rightarrow \Delta \text{ imply } \vdash_{\mathbf{G}_{\mathrm{a}}} \Gamma \Rightarrow \Delta'.$$

Proof. We prove that the axiom is monotonic and each deduction rule preserves the monotonicity.

Assume that $\Gamma \cap \Delta \neq \emptyset$ or incon(Γ) or incon(Δ). Then, for any formula sets Γ' and Δ' with $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$,

$$\Gamma' \cap \Delta \neq \emptyset \text{ or incon}(\Gamma') \text{ or incon}(\Delta)$$
$$\Gamma \cap \Delta' \neq \emptyset \text{ or incon}(\Gamma) \text{ or incon}(\Delta')$$
$$\Gamma' \cap \Delta' \neq \emptyset \text{ or incon}(\Gamma') \text{ or incon}(\Delta').$$

To show that $(\Rightarrow\wedge_1^L)$ preserves the monotonicity of Γ, assume that $\Gamma, A_1 \Rightarrow \Delta$ is monotonic with respect to Γ. By $(\Rightarrow\wedge_1^L)$, from $\Gamma, A_1 \Rightarrow \Delta$ we infer $\Gamma, A_1 \wedge A_2 \Rightarrow \Delta$. Then, for any $\Gamma' \supseteq \Gamma, \Gamma', A_1 \Rightarrow \Delta$; and by $(\Rightarrow\wedge_1^L)$, from $\Gamma', A_1 \Rightarrow \Delta$, infer $\Gamma', A_1 \wedge A_2 \Rightarrow \Delta$. Hence, $\Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ implies $\Gamma', A_1 \wedge A_2 \Rightarrow \Delta$, that is, $\Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ is monotonic with respect to Γ.

To show that $(_\Rightarrow\wedge_1^L)$ preserves the nonmonotonicity of Γ, assume that $\Gamma, A_1 \Rightarrow \Delta$ is nonmonotonic with respect to Γ. By $(_\Rightarrow\wedge_1^L)$, from $\Gamma, A_1 \Rightarrow \Delta$ we infer $\Gamma, A_1 \wedge A_2 \Rightarrow \Delta$. Then, for some $\Gamma' \supseteq \Gamma, \Gamma, A_1 \Rightarrow \Delta$ may not imply $\Gamma', A_1 \Rightarrow \Delta$; and by $(_\Rightarrow\wedge^L), \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ may not imply $\Gamma', A_1 \wedge A_2 \Rightarrow \Delta$, that is, $\Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ is nonmonotonic with respect to Γ.

Similar to show that other deduction rules preserves the monotonicity and nonmonotonicity with respect to Γ and Δ.

□

A co-sequent $\Gamma \mapsto \Delta$ is $\mathbf{G}_\mathrm{a}$-valid, denoted by $\models_{\mathbf{G}_\mathrm{a}} \Gamma \mapsto \Delta$, if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for every $B \in \Delta, v(B) = 0$.

Proposition 3.4. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}_\mathrm{a}} \Gamma \mapsto \Delta$ if and only if $\Gamma \cap \Delta = \emptyset \& \mathrm{con}(\Gamma) \& \mathrm{con}(\Delta)$.

□

Gentzen deduction system $\mathbf{G}_\mathrm{a}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_\mapsto)\ \frac{\Gamma \cap \Delta = \emptyset \& \mathrm{con}(\Gamma) \& \mathrm{con}(\Delta)}{\Gamma \mapsto \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(_\mapsto\neg\neg^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, \neg\neg A_1 \mapsto \Delta} \qquad (_\mapsto\neg\neg^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto \neg\neg B_1, \Delta}$$

$$(_\mapsto\wedge^L)\ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (_\mapsto\wedge_1^R)\ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_\mapsto\wedge_2^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_\mapsto\vee_1^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (_\mapsto\vee^R)\ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(_\mapsto\vee_2^L)\ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta}$$

$$(_\mapsto\neg\wedge_1^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (_\mapsto\neg\wedge^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta \quad \Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(_\mapsto\neg\wedge_2^L)\ \frac{\Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta}$$

$$(_\mapsto\neg\vee^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta \quad \Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \vee A_2) \mapsto \Delta} \qquad (_\mapsto\neg\vee_1^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

$$(_\mapsto\neg\vee_2^R)\ \frac{\Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

Theorem 3.5(Soundness and completeness theorem). For any co-sequent $\Gamma \mapsto \Delta$, $\vdash_{\mathbf{G}_\mathrm{a}} \Gamma \mapsto \Delta$ if and only if $\models_{\mathbf{G}_\mathrm{a}} \Gamma \mapsto \Delta$.

□

Theorem 3.6(Monotonicity theorem). $\mathbf{G}_\mathrm{a}$ is nonmonotonic in both Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_{\rm a}} \Gamma \mapsto \Delta \text{ may not imply } \vdash_{\mathbf{G}_{\rm a}} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_{\rm a}} \Gamma \mapsto \Delta \text{ may not imply } \vdash_{\mathbf{G}_{\rm a}} \Gamma \mapsto \Delta'.$$

Proof. We prove that the axiom is nonmonotonic and each deduction rule preserves the monotonicity.

Assume that $\Gamma \cap \Delta = \emptyset \& \text{con}(\Gamma) \& \text{con}(\Delta)$. There is a superset $\Gamma' \supseteq \Gamma$ such that Γ' is inconsistent; and there is a superset $\Delta' \supseteq \Delta$ such that Δ' is inconsistent. Hence, $\mathbf{G}_{\rm a}$ is nonmonotonic in both Γ and Δ.

Similar to show that the deduction rules preserves the monotonicity and nonmonotonicity with respect to Γ and Δ.

□

4. Gentzen deduction systems $\mathbf{G}^{\rm b}$ and $\mathbf{G}_{\rm b}$

A sequent $\Gamma \Rightarrow \Delta$ is $\mathbf{G}^{\rm b}$-valid, denoted by $\models_{\mathbf{G}^{\rm b}} \Gamma \Rightarrow \Delta$, if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for every $B \in \Delta, v(B) = 1$.

Proposition 4.1. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}^{\rm b}} \Gamma \Rightarrow \Delta$ if and only if incon(Γ) or $\Delta \subseteq \Gamma$.

Proof. If incon(Γ) or $\Delta \subseteq \Gamma$ then for any assignment $v, v \models \Gamma \Rightarrow \Delta$.

Conversely, assume that con$(\Gamma) \& \Delta \not\subseteq \Gamma$. Let $l \in \Delta - \Gamma$. Define an assignment v such that for any propositional variable p,

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma \\ 1 & \text{if } p = \neg l \\ 0 & \text{otherwise.} \end{cases}$$

Then, v is well-defined, and $v \not\models \Gamma \Rightarrow \Delta$.

□

Gentzen deduction system $\mathbf{G}^{\rm b}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\Rightarrow}) \ \frac{\text{incon}(\Gamma) \text{ or } \Delta \subseteq \Gamma}{\Gamma \Rightarrow \Delta},$$

where l is a literal.

• Deduction rules:

$$(\Rightarrow\neg\neg^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, \neg\neg A_1 \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow \neg\neg B_1, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\vee_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\neg\wedge^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta \quad \Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \wedge A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\wedge_1^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\wedge_2^R)\ \frac{\Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\vee_1^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta \quad \Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

$$(\Rightarrow\neg\vee_2^L)\ \frac{\Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta}$$

Theorem 4.2(Soundness and completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$, $\vdash_{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta$ if and only if $\models_{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta$.

□

Theorem 4.3(Monotonicity theorem). $\mathbf{G}^{\mathrm{b}}$ is monotonic in Γ and nonmonotonic in Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta \text{ imply } \vdash_{\mathbf{G}^{\mathrm{b}}} \Gamma' \Rightarrow \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta \text{ may not imply } \vdash^{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta'.$$

□

A co-sequent $\Gamma \mapsto \Delta$ is $\mathbf{G}_{\mathrm{b}}$-valid, denoted by $\models_{\mathbf{G}_{\mathrm{b}}} \Gamma \mapsto \Delta$, if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 0$.

Proposition 4.4. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}_{\mathrm{b}}} \Gamma \mapsto \Delta$ if and only if $\mathrm{con}(\Gamma)\&\Delta \not\subseteq \Gamma$.

Proof. $\models_{\mathbf{G}_{\mathrm{b}}} \Gamma \mapsto \Delta$ iff $\not\models_{\mathbf{G}^{\mathrm{b}}} \Gamma \Rightarrow \Delta$, iff $\neg(\mathrm{incon}(\Gamma)$ or $\Delta \subseteq \Gamma)$, iff $\neg\mathrm{incon}(\Gamma)\&\neg(\Delta \subseteq \Gamma)$, iff $\mathrm{con}(\Gamma)\&\Delta \not\subseteq \Gamma$.

□

Gentzen deduction system $\mathbf{G}_{\mathrm{b}}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\mapsto})\ \frac{\mathrm{con}(\Gamma)\&\Delta \not\subseteq \Gamma}{\Gamma \mapsto \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(\mapsto\neg\neg^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, \neg\neg A_1 \mapsto \Delta} \qquad (\mapsto\neg\neg^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto \neg\neg B_1, \Delta}$$

$$(\mapsto\wedge^L)\ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge_1^R)\ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\wedge_2^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee_1^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee^R)\ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee_2^L)\ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta}$$

$$(\mapsto\neg\wedge_1^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (\mapsto\neg\wedge^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta \quad \Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(\mapsto\neg\wedge_2^L)\ \frac{\Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta}$$

$$(\mapsto\neg\vee^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta \quad \Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \vee A_2) \mapsto \Delta} \qquad (\mapsto\neg\vee_1^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

$$(\mapsto\neg\vee_2^R)\ \frac{\Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

Theorem 4.5(Soundness and completeness theorem). For any co-sequent $\Gamma \mapsto \Delta$, $\vdash_{\mathbf{G}_\mathrm{b}} \Gamma \mapsto \Delta$ if and only if $\models_{\mathbf{G}_\mathrm{b}} \Gamma \mapsto \Delta$.

□

Theorem 4.6(Monotonicity theorem). $\mathbf{G}^\mathrm{b}$ is nonmonotonic in Γ and monotonic in Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_\mathrm{b}} \Gamma \mapsto \Delta \text{ may not imply } \vdash_{\mathbf{G}_\mathrm{b}} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_\mathrm{b}} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_\mathrm{b}} \Gamma \mapsto \Delta'.$$

□

5. Gentzen deduction systems $\mathbf{G}^\mathrm{c}$ and $\mathbf{G}_\mathrm{c}$

A sequent $\Gamma \Rightarrow \Delta$ is $\mathbf{G}^\mathrm{c}$-valid, denoted by $\models_{\mathbf{G}^\mathrm{c}} \Gamma \Rightarrow \Delta$, if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 0$.

Proposition 5.1. Let Γ and Δ be sets of literals. $\models_{\mathbf{G}^\mathrm{c}} \Gamma \Rightarrow \Delta$ if and only if incon(Γ) or $\Gamma \cap \neg\Delta \neq \emptyset$ or incon($\neg\Delta$), where $\neg\Delta = \{\neg l : l \in \Delta\}$.

□

Gentzen deduction system $\mathbf{G}^\mathrm{c}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_\Rightarrow)\ \frac{\text{incon}(\Gamma) \text{ or } \Gamma \cap \neg\Delta \neq \emptyset \text{ or incon}(\neg\Delta)}{\Gamma \Rightarrow \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(\Rightarrow\neg\neg^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, \neg\neg A_1 \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow \neg\neg B_1, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_2^L)\ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\neg\wedge^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta \quad \Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \wedge A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\wedge^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta \quad \Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\vee_1^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee_1^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

$$(\Rightarrow\neg\vee_2^L)\ \frac{\Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee_2^R)\ \frac{\Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

Theorem 5.2(Soundness and completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$,

$$\vdash_{\mathbf{G}^c} \Gamma \Rightarrow \Delta \text{ iff } \models_{\mathbf{G}^c} \Gamma \Rightarrow \Delta.$$

□

Theorem 5.3(Monotonicity theorem). $\mathbf{G}^c$ is monotonic in Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_c} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta'.$$

□

A co-sequent $\Gamma \mapsto \Delta$ is $\mathbf{G}_c$-valid, denoted by $\models_{\mathbf{G}_c} \Gamma \mapsto \Delta$, if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for every $B \in \Delta, v(B) = 1$.

Proposition 5.4. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}_c} \Gamma \mapsto \Delta$ if and only if $\mathrm{con}(\Gamma) \& \Gamma \cap \neg\Delta = \emptyset \& \mathrm{con}(\neg\Delta)$.

□

Gentzen deduction system $\mathbf{G}_c$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\mapsto})\ \frac{\mathrm{con}(\Gamma) \& \Gamma \cap \neg\Delta = \emptyset \& \mathrm{con}(\neg\Delta)}{\Gamma \mapsto \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(\mapsto\neg\neg^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, \neg\neg A_1 \mapsto \Delta} \qquad (\mapsto\neg\neg^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto \neg\neg B_1, \Delta}$$

$$(\mapsto\wedge^L)\ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge^R)\ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee^L_1)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee^R_1)\ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee^L_2)\ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee^R_2)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\neg\wedge^L_1)\ \frac{\Gamma, \neg A_1 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (\mapsto\neg\wedge^R_1)\ \frac{\Gamma \mapsto \neg B_1, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(\mapsto\neg\wedge^L_2)\ \frac{\Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (\mapsto\neg\wedge^R_2)\ \frac{\Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(\mapsto\neg\vee^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta \quad \Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \vee A_2) \mapsto \Delta} \qquad (\mapsto\neg\vee^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta \quad \Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

Theorem 5.5(Soundness and completeness theorem). For any co-sequent $\Gamma \mapsto \Delta$,

$$\vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta \text{ iff } \models_{\mathbf{G}_c} \Gamma \mapsto \Delta.$$

□

Theorem 5.6(Monotonicity theorem). $\mathbf{G}_c$ is nonmonotonic in Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_c} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_c} \Gamma \mapsto \Delta'.$$

□

6. Gentzen deduction systems $\mathbf{G}^d$ and $\mathbf{G}_d$

A sequent $\Gamma \Rightarrow \Delta$ is $\mathbf{G}^d$-valid, denoted by $\models_{\mathbf{G}^d} \Gamma \Rightarrow \Delta$, if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for every $B \in \Delta, v(B) = 0$.

Proposition 6.1. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}^d} \Gamma \Rightarrow \Delta$ if and only if $\text{incon}(\Gamma)$ or $\neg\Delta \subseteq \Gamma$.

□

Gentzen deduction system $\mathbf{G}^d$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\Rightarrow})\ \frac{\text{incon}(\Gamma) \text{ or } \neg\Delta \subseteq \Gamma}{\Gamma \Rightarrow \Delta},$$

where l is a literal.

• Deduction rules:

$$(\Rightarrow\neg\neg^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, \neg\neg A_1 \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow \neg\neg B_1, \Delta}$$

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_2^L)\ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(\Rightarrow\neg\wedge^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta \quad \Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \wedge A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\wedge^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta \quad \Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \wedge B_2), \Delta}$$

$$(\Rightarrow\neg\vee_1^L)\ \frac{\Gamma, \neg A_1 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee_1^R)\ \frac{\Gamma \Rightarrow \neg B_1, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

$$(\Rightarrow\neg\vee_2^L)\ \frac{\Gamma, \neg A_2 \Rightarrow \Delta}{\Gamma, \neg(A_1 \vee A_2) \Rightarrow \Delta} \qquad (\Rightarrow\neg\vee_2^R)\ \frac{\Gamma \Rightarrow \neg B_2, \Delta}{\Gamma \Rightarrow \neg(B_1 \vee B_2), \Delta}$$

Theorem 6.2(Soundness and completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$,

$$\vdash_{\mathbf{G}^{\mathrm{d}}} \Gamma \Rightarrow \Delta \text{ iff } \models_{\mathbf{G}^{\mathrm{d}}} \Gamma \Rightarrow \Delta.$$

□

Theorem 6.3(Monotonicity theorem). $\mathbf{G}^{\mathrm{d}}$ is monotonic in Γ and nonmonotonic in Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_{\mathrm{d}}} \Gamma \Rightarrow \Delta \text{ imply } \vdash_{\mathbf{G}_{\mathrm{d}}} \Gamma' \Rightarrow \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_{\mathrm{d}}} \Gamma \Rightarrow \Delta \text{ may not imply } \vdash_{\mathbf{G}_{\mathrm{d}}} \Gamma \Rightarrow \Delta'.$$

□

A co-sequent $\Gamma \mapsto \Delta$ is $\mathbf{G}_{\mathrm{d}}$-valid, denoted by $\models_{\mathbf{G}_{\mathrm{d}}} \Gamma \mapsto \Delta$, if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1$.

Proposition 6.4. Let Γ, Δ be sets of literals. $\models_{\mathbf{G}_{\mathrm{d}}} \Gamma \mapsto \Delta$ if and only if $\mathrm{con}(\Gamma) \& \neg\Delta \not\subseteq \Gamma$.

□

Gentzen deduction system $\mathbf{G}_{\mathrm{d}}$ consists of the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A}_{\mapsto})\ \frac{\mathrm{con}(\Gamma) \& \neg\Delta \not\subseteq \Gamma}{\Gamma \mapsto \Delta},$$

where Δ, Γ are sets of literals.

• Deduction rules:

$$(\mapsto\neg\neg^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, \neg\neg A_1 \mapsto \Delta} \qquad (\mapsto\neg\neg^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto \neg\neg B_1, \Delta}$$

$$(\mapsto\wedge^L)\ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge^R)\ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee_1^L)\ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_1^R)\ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee_2^L)\ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_2^R)\ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\neg\wedge_1^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (\mapsto\neg\wedge_1^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(\mapsto\neg\wedge_2^L)\ \frac{\Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \wedge A_2) \mapsto \Delta} \qquad (\mapsto\neg\wedge_2^R)\ \frac{\Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \wedge B_2), \Delta}$$

$$(\mapsto\neg\vee^L)\ \frac{\Gamma, \neg A_1 \mapsto \Delta \quad \Gamma, \neg A_2 \mapsto \Delta}{\Gamma, \neg(A_1 \vee A_2) \mapsto \Delta} \qquad (\mapsto\neg\vee^R)\ \frac{\Gamma \mapsto \neg B_1, \Delta \quad \Gamma \mapsto \neg B_2, \Delta}{\Gamma \mapsto \neg(B_1 \vee B_2), \Delta}$$

Theorem 6.5(Soundness and completeness theorem). For any co-sequent $\Gamma \mapsto \Delta$,

$$\vdash_{\mathbf{G}_\mathrm{d}} \Gamma \mapsto \Delta \text{ iff } \models_{\mathbf{G}_\mathrm{d}} \Gamma \mapsto \Delta.$$

□

Theorem 6.6(Monotonicity theorem). $\mathbf{G}^\mathrm{d}$ is nonmonotonic in both Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{\mathbf{G}_\mathrm{d}} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_\mathrm{d}} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{\mathbf{G}_\mathrm{d}} \Gamma \mapsto \Delta \text{ imply } \vdash_{\mathbf{G}_\mathrm{d}} \Gamma \mapsto \Delta'.$$

□

7. Conclusions

In this paper we proved that $\mathbf{G}^\mathrm{a}, \mathbf{G}^\mathrm{b}, \mathbf{G}^\mathrm{c}, \mathbf{G}^\mathrm{d}, \mathbf{G}_\mathrm{a}, \mathbf{G}_\mathrm{b}, \mathbf{G}_\mathrm{c}, \mathbf{G}_\mathrm{d}$ are sound and complete, and their monotonicity given in the following table:

system	precondition	mono in Γ	mono in Δ
$\mathbf{G}^\mathrm{a}$	$\Gamma \cap \Delta \neq \emptyset$ or incon(Γ) or incon(Δ)	Y	Y
$\mathbf{G}_\mathrm{a}$	$\Gamma \cap \Delta = \emptyset$&con($\Gamma$)&con($\Delta$)	N	N
$\mathbf{G}^\mathrm{b}$	$\Delta \subseteq \Gamma$ or incon(Γ)	Y	N
$\mathbf{G}_\mathrm{b}$	$\Delta \not\subseteq \Gamma$&con($\Gamma$)	N	Y
$\mathbf{G}^\mathrm{c}$	$\Gamma \cap \neg\Delta \neq \emptyset$ or incon(Γ) or incon($\neg\Delta$)	Y	Y
$\mathbf{G}_\mathrm{c}$	$\Gamma \cap \neg\Delta = \emptyset$&con($\Gamma$)&con($\neg\Delta$)	N	N
$\mathbf{G}^\mathrm{d}$	$\neg\Delta \subseteq \Gamma$ or incon(Γ)	Y	N
$\mathbf{G}_\mathrm{d}$	$\neg\Delta \not\subseteq \Gamma$&con($\Gamma$)	N	Y

where $\neg\Delta = \{\neg A : A \in \Delta\}$. Hence, $\neg\Delta$ is consistent if and only if Δ is consistent.

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