1. Proof of bilinear state transition dynamics

The proof will be completed in two stages. First we show that the transition function of A, $f: X \times E \to 2^X$, can be formulated in matrix form as $f: \Delta_n \times \Delta_m \to 2^{\Delta_n}$, where

$$f(\delta_n^i, \delta_m^j) = F \ltimes \delta_m^j \ltimes \delta_n^i, \tag{1}$$

or

$$f(\delta_n^i, \delta_m^j) = \tilde{F} \ltimes \delta_n^i \ltimes \delta_m^j, \tag{2}$$

in which $\tilde{F} = F \ltimes W_{[n,m]}$; δ_m^j and δ_n^i are the vector forms of e_j and x_i, i=1,2,$\cdots$,n, j=1,2,$\cdots$,m, respectively; 2^{Δ_n} is the vector form of 2^X.

Assume that input e_j moves the machine A from state x_i to state x_k, $1 \le k \le n$, i.e., $f(x_i, e_j) = x_k$. According to the definition of $F_{i(s,t)}$, we know that the k-th element of $\mathrm{col}_i(F_j)$ equal to 1, other elements equal to 0. Thus we get $\mathrm{col}_i(F_j) = \delta_n^k$.

Moreover, a straightforward computing indicates that

$$F \ltimes \delta_m^j \ltimes \delta_n^i = \delta_n^k.$$

In addition, δ_n^k is the vector form of x_k, we know that

$$f(x_i, e_j) = F \ltimes \delta_m^j \ltimes \delta_n^i.$$

Equation (1) thus holds. Equation (2) can be proofed immediately by the pseudo commutative law of vectors given in the body of the letter.

Next, we prove the conclusion of the bilinear state transition dynamics. According to the dynamics that a finite automaton reads an input string $e = e_1 e_2 \cdots e_t$, we have

$$\begin{aligned}
x(t+1) &= f(x_i, e) \\
&= f(\cdots f(f(f(x_i, e_1), e_2), e_3), \cdots, e_t) \\
&= f(\cdots f(f(\tilde{F} \ltimes \delta_n^i \ltimes \delta_m^1, e_2), e_3), \cdots, e_t) \\
&= f(\cdots f(\tilde{F} \ltimes \tilde{F} \ltimes \delta_n^i \ltimes \delta_m^1 \ltimes \delta_m^2, e_3), \cdots, e_t) \\
&\cdots \\
&= \underbrace{\tilde{F} \ltimes \cdots \ltimes \tilde{F}}_{t} \ltimes \delta_n^i \ltimes \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^t \\
&= \tilde{F}^t \ltimes \delta_n^i \ltimes u(t).
\end{aligned}$$

From the above analysis, we then get the conclusion. The proof is completed.

2. Proof of necessary and sufficient conditions of connection of two states

(Sufficiency). It is easy to know by a straightforward computation that all columns of C_i^k belong to the set of columns of the unit matrix I_n, that is $\mathrm{col}_i(C_i^k) \in \Delta_n, (i$=1,2,$\cdots$,$m^t)$.

If $\delta_n^q \in \mathrm{col}(C_i^k)$, say, $\delta_n^q = \mathrm{col}_h(C_i^k)$, then $u(k) = \delta_{m^k}^h$ is a solution of the following equation with x being an unknown variable.

$$C_i^k \ltimes x = \delta_n^q. \tag{3}$$

On the other hand, we note that $u(k) = \delta_{m^k}^h$ can be treated as a semi-tensor product of the vector forms of k inputs, say, $e_1, \cdots, e_k$, that is, $u(k) = \delta_m^1 \ltimes \cdots \ltimes \delta_m^k$.

Comparing (3) with the conclusion of bilinear state transition dynamics, we find that $e = e_1 e_2 \cdots e_k$ is a input sequence that makes A reach x_j from x_i. This demonstrates that the state x_i is connected to the state x_j with a path of length k.

Besides, we can further obtain the input sequence through solving the following k-ary equation

$$\ltimes_{i=1}^{k} e_i = \delta_{m^k}^{h}, \tag{4}$$

in which $e_i \in \Delta_m$.

(Necessity) If the state $x_i := \delta_n^p$ is connected to the state $x_j := \delta_n^q$ by an input sequence of length k, say, $e = e_1 e_2 \cdots e_k$, according to Theorem 1, we have

$$\begin{aligned} x_j &:= \delta_n^q \\ &= \tilde{F}^k \ltimes \delta_n^p \ltimes u(k). \\ &= C_i^k \ltimes u(k), \end{aligned}$$

in which $u(k) = \ltimes_{i=1}^{k} \delta_m^i = \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^k$, δ_m^j is the vector form of e_j, $j=1,2,\cdots,k$.

Since only one element of $u(k)$ is 1, others are 0s, we get

$$x_j := \delta_n^q \in \mathrm{col}(C_i^k). \tag{5}$$

That is just $\delta_n^q \in \mathrm{col}(C_i^k)$. The necessity is then obtained. Thus, the proof is completed.

3. Proof of Recognition criteria of understanding regular language

We first show that C^s is a logic matrix for an arbitrary positive integer s, after which the denotation of $K_U^{C^s}$ makes sense.

We know from the bilinear state transition dynamics that

$$C^s = \tilde{F}^s \ltimes \delta_n^1. \tag{6}$$

$F_{i(s,t)}$ indicates that F is a logic matrix of dimension $n \times mn$ for a deterministic finite automaton. On the other hand, $\tilde{F} = F \ltimes W_{[n,m]}$, where $W_{[n,m]}$ is a swap matrix whose function is just to change the positions of the columns of F, hence $\tilde{F}$ is still a logic matrix with the same dimension as F.

It is easy to know that the semi-tensor product of two logic matrices is a logic matrix. For a special case, the semi-tensor product of s identical logic matrices, $\tilde{F}^s$, is a logic matrix with dimension $n \times nm^s$. In fact, computing $\tilde{F}^s \ltimes \delta_n^1$ tells us that C^s is indeed a logic matrix of dimension $n \times m^s$.

(Necessity) If the sentence $e = e_1 e_2 \cdots e_s$ is acceptable by A, by Theorem 1’, we have

$$x(s+1) = C^s \ltimes \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^s \in U.$$

Note that C^s is of dimension $n \times m^s$ and $u = \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^s$ is a vector of dimension $m^s \times 1$ with only element being 1 and others being 0s. Let $u = \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^s = \delta_{m^s}^r$ without loss of generality, i.e., $\pi(e) = r$. Recall the definition of the semi-tensor product of matrices and we know that the semi-tenor product operation of $C^s \ltimes u$ reduces to the ordinary product of matrices. Therefore it is easy to know that the final state $x(s+1)$ is just the r-th column of C^s, i.e.,

$$x(s+1) = \mathrm{col}_r(C^s). \tag{7}$$

We then get that

$$r \in K_{s+1}^{C^s},$$

that is

$$\pi(e) \in K_{s+1}^{C^s}.$$

Since $x(s+1)$ is an acceptable state, i.e., $x(s+1) \in U$, according to the definition of K_S^L, the following holds naturally.

$$\pi(e) \in K_U^{C^s}.$$

Then necessity is then obtained.

(Sufficiency) For a given sentence $e = e_1 e_2 \cdots e_s$, assume that $\pi(e) \in K_U^{C^s}$, say, $\pi(e) \in K_i^{C^s}$ $(x_i \in U)$, which implies according to the definition of K_i^L that the $\pi(e)$-th column of C^s is the vector form of an acceptable state, say, x_j. Again recall the definition of the semi-tensor product of matrices and that C^s is of dimension $n \times m^s$ and $\delta_{m^s}^{\pi(e)}$ is a vector of dimension $m^s \times 1$ with the $\pi(e)$-th element being 1 and others being 0s, in this case, the semi-tenor product operation of $C^s \ltimes \delta_{m^s}^{\pi(e)}$ reduces to the ordinary product of matrices, which implies that

$$C^s \ltimes \delta_{m^s}^{\pi(e)} = x_j \in U \ .$$

On the other hand, we know from Definition 5 that

$$\delta_{m^s}^{\pi(e)} = \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^s \ .$$

Then we get

$$C^s \ltimes \delta_m^1 \ltimes \delta_m^2 \ltimes \cdots \ltimes \delta_m^s = x_j \in U \ ,$$

which indicates that the sentence $e = e_1 e_2 \cdots e_s$ can move A to the accepted state x_j from the initial state, in other words, $e = e_1 e_2 \cdots e_s$ can be accepted by A. The sufficiency is proved.