Online Resource

A Proof of Lemma 1.1

Lemma 1.1. For any two real numbers s and t with $s \geq t > 1$, we have $\log^t s \leq s \cdot t^{O(t)}$.

Proof. For the case where $t < \frac{\log s}{\log\log s}$, it can be seen that $\log^t s < \log^{\frac{\log s}{\log\log s}} s = s$. For the case where $t \geq \frac{\log s}{\log\log s}$, we have $t \geq \sqrt{\log s}$, which implies that $\log^t s \leq t^{O(t)}$. Putting together, we have $\log^t s \leq s \cdot t^{O(t)}$, as desired. □

B Proof of Lemma 2.1

We first introduce some notations to help with analysis. Let $\mathcal{R}_0 = \mathbb{H}_0 = \emptyset$. Given an integer $i \in [720k\varepsilon^{-3}+1]$, let $\mathcal{R}_i$ denote the set of the clients identified by Algorithm 1 after its i-th iteration, and define $\mathbb{H}_i = \{C_j^* \in \mathbb{C}^* : d(f_j^*, \mathcal{R}_i) \leq (1+\frac{\varepsilon}{2})r_j\}$, where $r_j = opt_j/|C_j^*|$. Given a client set $C_j^* \in \mathbb{C}^*$ and a real number $u > 0$, define $B(C_j^*, u) = \{c \in C_j^* : d(c, f_j^*) \leq u \cdot r_j\}$. To prove Lemma 2.1, we will show the following invariant $\kappa(i)$ for each integer $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$, which says that either $\mathcal{R}_i$ is a desired subset of $\mathcal{R}$ described in Lemma 2.1, or inequality $|\mathbb{H}_{i+1}| > |\mathbb{H}_i|$ holds with high probability.

For each integer $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$, let $\kappa(i)$ be the invariant saying that $\sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}_i) \leq (1+\varepsilon)opt$ or $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq \varepsilon^3/360$. We now focus on showing that $\kappa(i)$ remains true for each $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$. We first give a valuable lower bound on the probability of $|\mathbb{H}_{i+1}| > |\mathbb{H}_i|$ for each $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$.

Lemma 4.1. We have

$$\Pr[|\mathbb{H}_1| > |\mathbb{H}_0|] \geq \sum_{j=1}^{k^*} |B(C_j^*, 1+\frac{\varepsilon}{2})|/|C|$$

and

$$\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq \sum_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} d^{\text{sum}}(B(C_j^*, 1+\frac{\varepsilon}{2}), \mathcal{R}_i)/d^{\text{sum}}(C, \mathcal{R}_i)$$

for each $i \in [720k\varepsilon^{-3}]$ and $\varepsilon \in (0,1]$.

Proof. Given an integer $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$ and a client set $C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i$, if Algorithm 1 adds a client c from $B(C_j^*, 1+\frac{\varepsilon}{2})$ to $\mathcal{R}_{i+1}$, then we have $C_j^* \in \mathbb{H}_{i+1} \setminus \mathbb{H}_i$ and $|\mathbb{H}_{i+1}| > |\mathbb{H}_i|$ due to the definitions of $\mathbb{H}_i$ and $B(C_j^*, 1+\frac{\varepsilon}{2})$. Consequently, the probability of $|\mathbb{H}_{i+1}| > |\mathbb{H}_i|$ is no less than the probability of selecting a client from $\bigcup_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} B(C_j^*, 1+\frac{\varepsilon}{2})$ in the $(i+1)$-th iteration of Algorithm 1. We can see that Algorithm 1 selects a client $c \in \bigcup_{C_j^* \in \mathbb{C}^*} B(C_j^*, 1+\frac{\varepsilon}{2})$ and lets $\mathcal{R}_1 = \{c\}$ with probability $\sum_{j=1}^{k^*} |B(C_j^*, 1+\frac{\varepsilon}{2})|/|C|$, and for each $i \in [720k\varepsilon^{-3}]$, it selects clients from $\bigcup_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} B(C_j^*, 1+\frac{\varepsilon}{2})$ in the $(i+1)$-th iteration with probability $\sum_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} d^{\text{sum}}(B(C_j^*, 1+\frac{\varepsilon}{2}), \mathcal{R}_i)/d^{\text{sum}}(C, \mathcal{R}_i)$. Thus, Lemma 4.1 is true. □

The following result says that $B(C_j^*, u)$ accounts for a large proportion of C_j^* for some $u \geq 1$ and each $C_j^* \in \mathbb{C}^*$.

Lemma 4.2. We have $|B(C_j^*, u)| \geq (1-\frac{1}{u})|C_j^*|$ for each $u \geq 1$ and $C_j^* \in \mathbb{C}^*$.

Proof. Observe that

$$\begin{aligned} d(C_j^* \setminus B(C_j^*, u), f_j^*) &> u|C_j^* \setminus B(C_j^*, u)| r_j \\ &= u|C_j^* \setminus B(C_j^*, u)| \frac{opt_j}{|C_j^*|}, \end{aligned} \tag{4}$$

where the first step follows from the definition of $B(C_j^*, u)$, and the second step is due to the definition of r_j. Moreover, it is the case that

$$d(C_j^* \setminus B(C_j^*, u), f_j^*) \leq d(C_j^*, f_j^*) = opt_j. \tag{5}$$

We have $|C_j^* \setminus B(C_j^*, u)| < \frac{1}{u}|C_j^*|$ using inequalities (4) and (5), implying that $|B(C_j^*, u)| \geq (1-\frac{1}{u})|C_j^*|$. Thus, Lemma 4.2 is true. □

Observe that

$$\begin{aligned} \Pr[|\mathbb{H}_1| > |\mathbb{H}_0|] &\geq \frac{1}{|C|}\sum_{j=1}^{k^*} |B(C_j^*, 1+\frac{\varepsilon}{2})| \\ &= \frac{\sum_{j=1}^{k^*} |B(C_j^*, 1+\frac{\varepsilon}{2})|}{\sum_{j=1}^{k^*} |C_j^*|} \\ &\geq \min_{j \in [k^*]} \frac{|B(C_j^*, 1+\frac{\varepsilon}{2})|}{|C_j^*|} \\ &\geq 1 - \frac{1}{1+\varepsilon/2} > \frac{\varepsilon^3}{360}, \end{aligned} \tag{6}$$

where the first step follows from Lemma 4.1, the fourth step is due to Lemma 4.2, and the last step is derived from the fact that $\varepsilon \in (0,1]$. This implies that $\kappa(i)$ holds for $i = 0$.

We now show the correctness of $\kappa(i)$ for $i > 0$. Given an integer $i \in [720k\varepsilon^{-3}]$, we consider the following two cases: (1) $\sum_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) > \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$, and (2) $\sum_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$.

Given a client set $C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i$, let g_j be the client from $\mathcal{R}_i$ nearest to f_j^*, and define $d_j = d(g_j, f_j^*)$ and $v_j = d_j/r_j$. Given a client $c \in C$, let $g(c)$ denote the client from $\mathcal{R}_i$ nearest to c. We first consider case (1), where we show that inequality $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq \varepsilon^3/360$ holds.

Lemma 4.3. We have $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] > \varepsilon^3/360$ if $\sum_{C_j^* \in \mathbb{C}^* \setminus \mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) > \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$ for each $i \in [720k\varepsilon^{-3}]$ and $\varepsilon \in (0,1]$.

Proof. Let C_j^* denote an arbitrary client set from $\mathbb{C}^*\backslash\mathbb{H}_i$. For each $u \geq 1$, it is the case that

$$\begin{aligned} d^{\text{sum}}(B(C_j^*, u), \mathcal{R}_i) &= \sum_{c \in B(C_j^*, u)} d(c, g(c)) \\ &\geq \sum_{c \in B(C_j^*, u)} (d(g(c), f_j^*) - d(c, f_j^*)) \\ &\geq \sum_{c \in B(C_j^*, u)} (d(g_j, f_j^*) - d(c, f_j^*)) \\ &\geq \sum_{c \in B(C_j^*, u)} (d_j - u \cdot r_j) \\ &= |B(C_j^*, u)|(d_j - u \cdot r_j) \\ &\geq (1 - \frac{1}{u})|C_j^*|(d_j - u \cdot r_j), \end{aligned} \tag{7}$$

where the first step is derived from the definition of $g(c)$, the second step is derived from triangle inequality, the third step is due to the definition of g_i, the fourth step is due to the definitions of $B(C_j^*, u)$ and d_j, and the last step follows from Lemma 4.2. Moreover, we have

$$\begin{aligned} d^{\text{sum}}(C_j^*, \mathcal{R}_i) &\leq d(C_j^*, g_j) \leq d(C_j^*, f_j^*) + |C_j^*|d(f_j^*, g_j) \\ &= |C_j^*|(r_j + d_j), \end{aligned} \tag{8}$$

where the first step is due to the definition of $d^{\text{sum}}(C_j^*, \mathcal{R}_i)$, the second step is due to triangle inequality, and the last step follows from the definitions of r_j and d_j.

Inequalities (7) and (8) imply that

$$\begin{aligned} \frac{d^{\text{sum}}(B(C_j^*, u), \mathcal{R}_i)}{d^{\text{sum}}(C_j^*, \mathcal{R}_i)} &\geq \frac{d_j - u \cdot r_j}{d_j + r_j}(1 - \frac{1}{u}) \\ &= \frac{v_j - u}{v_j + 1}(1 - \frac{1}{u}), \end{aligned} \tag{9}$$

where the last step is derived from the definition of v_j. Thus, it is the case that

$$\begin{aligned} \frac{d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)}{d^{\text{sum}}(C_j^*, \mathcal{R}_i)} &\geq \frac{d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{3}), \mathcal{R}_i)}{d^{\text{sum}}(C_j^*, \mathcal{R}_i)} \\ &\geq \frac{(v_j - 1 - \varepsilon/3)\varepsilon}{(1 + v_j)(3 + \varepsilon)}, \end{aligned} \tag{10}$$

where the first step is due to the fact that $B(C_j^*, 1 + \frac{\varepsilon}{3}), \mathcal{R}_i)$ is a subset of $B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)$, the second step follows from inequality (9). It can be seen that the right-hand side of inequality (10) increases monotonically with increasing value of v_j for the case where $v_j > 1$. Combining this with the fact that $v_j > 1 + \frac{\varepsilon}{2}$ (due to the fact that $C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i$ and the definitions of v_j and $\mathbb{H}_i$), we get

$$\frac{d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)}{d^{\text{sum}}(C_j^*, \mathcal{R}_i)} \geq \frac{(\varepsilon/2 - \varepsilon/3)\varepsilon}{(2 + \varepsilon/2)(3 + \varepsilon)} \geq \frac{\varepsilon^2}{60}, \tag{11}$$

where the last step follows from the fact that $\varepsilon \in (0, 1]$. Inequality (11) implies that

$$\begin{aligned} &\frac{\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)}{\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i)} \\ &\geq \min_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} \frac{d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)}{d^{\text{sum}}(C_j^*, \mathcal{R}_i)} \geq \frac{\varepsilon^2}{60}. \end{aligned} \tag{12}$$

Consequently, we have

$$\begin{aligned} \Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] &\geq \frac{\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(B(C_j^*, 1 + \frac{\varepsilon}{2}), \mathcal{R}_i)}{d^{\text{sum}}(C, \mathcal{R}_i)} \\ &> \frac{\varepsilon^*}{60} \cdot \frac{\varepsilon}{6} = \frac{\varepsilon^3}{360}, \end{aligned}$$

where the first step is due to Lemma 4.1, and the second step follows from the assumption that $\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) > \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$ and inequality (12). This implies that Lemma 4.3 is true. □

We now consider the case where inequality $\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$ holds. It is shown that the desired client set in Lemma 2.1 has been obtained for this case.

Lemma 4.4. For each $i \in [720k\varepsilon^{-3}]$ and $\varepsilon \in (0, 1]$, it is the case that $\sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}_i) \leq (1 + \varepsilon)opt$ if $\sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$.

Proof. Considering a client set $C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i$, we have

$$\begin{aligned} d(g_j, f_j^*) = \min_{c \in \mathcal{R}_i} d(c, f_j^*) &\leq \frac{1}{|C_j^*|} \sum_{c \in C_j^*} d(g(c), f_j^*) \\ &\leq \frac{1}{|C_j^*|} (\sum_{c \in C_j^*} d(g(c), c) + d(C_j^*, f_j^*)) \\ &= \frac{1}{|C_j^*|} (d^{\text{sum}}(C_j^*, \mathcal{R}_i) + opt_j), \end{aligned} \tag{13}$$

where the first step follows directly from the definition of g_j, the second step follows from the fact that $g(c) \in \mathcal{R}_i$ for each $c \in C_j^*$, the third step follows from triangle inequality, and the last step is derived from the fact that $g(c)$ denotes the client from $\mathcal{R}_i$ nearest to c. Thus, it is the case that

$$\begin{aligned} &\sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}_i) \\ &= \sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} |C_j^*| d(f_j^*, \mathcal{R}_i) + \sum_{C_j^* \in \mathbb{H}_i} |C_j^*| d(f_j^*, \mathcal{R}_i) \\ &\leq \sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} |C_j^*| d(f_j^*, \mathcal{R}_i) + (1 + \frac{\varepsilon}{2}) \sum_{C_j^* \in \mathbb{H}_i} opt_j \\ &= \sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} |C_j^*| d(f_j^*, g_j) + (1 + \frac{\varepsilon}{2}) \sum_{C_j^* \in \mathbb{H}_i} opt_j \\ &\leq \sum_{C_j^* \in \mathbb{C}^*\backslash\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) + (1 + \frac{\varepsilon}{2}) opt, \end{aligned} \tag{14}$$

where the second step is due to the definition of $\mathbb{H}_i$, and the last step is due to inequality (13) and the fact that $\sum_{j=1}^{k^*} opt_j = opt$.

We now show an upper bound on $\sum_{C_j^*\in\mathbb{C}^*\setminus\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i)$. Based on inequality $\sum_{C_j^*\in\mathbb{C}^*\setminus\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i)$, we have

$$\begin{aligned} d^{\text{sum}}(C, \mathcal{R}_i) &= \sum_{C_j^*\in\mathbb{C}^*\setminus\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) + \sum_{C_j^*\in\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \\ &\leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i) + \sum_{C_j^*\in\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i), \end{aligned}$$

which implies that

$$\begin{aligned} d^{\text{sum}}(C, \mathcal{R}_i) &\leq \frac{6}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) \\ &\leq \frac{6}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} d^{\text{min}}(C_j^*, \mathcal{R}_i) \\ &\leq \frac{6}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} (d(C_j^*, f_j^*) + |C_j^*| d(f_j^*, \mathcal{R}_i)) \\ &\leq \frac{6(\varepsilon/2+2)}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} opt_j \\ &= \frac{12+3\varepsilon}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} opt_j, \end{aligned} \tag{15}$$

where the second step follows from the definitions of $d^{\text{sum}}(C_j^*, \mathcal{R}_i)$ and $d^{\text{min}}(C_j^*, \mathcal{R}_i)$, the third step follows from triangle inequality, and the fourth step is due to the definition of $\mathbb{H}_i$. Consequently, we get

$$\begin{aligned} \sum_{C_j^*\in\mathbb{C}^*\setminus\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) &\leq \frac{\varepsilon}{6} d^{\text{sum}}(C, \mathcal{R}_i) \\ &\leq \frac{\varepsilon}{6} \cdot \frac{12+3\varepsilon}{6-\varepsilon} \sum_{C_j^*\in\mathbb{H}_i} opt_j \\ &\leq \frac{\varepsilon}{2} \sum_{C_j^*\in\mathbb{H}_i} opt_j \leq \frac{\varepsilon}{2} opt, \end{aligned} \tag{16}$$

where the first step is the assumption made by the lemma, the second step follows from inequality (15), and the last step follows from the fact that $\varepsilon \in (0, 1]$.

Using inequalities (14) and (16), we have

$$\begin{aligned} \sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}_i) &\leq \sum_{C_j^*\in\mathbb{C}^*\setminus\mathbb{H}_i} d^{\text{sum}}(C_j^*, \mathcal{R}_i) + (1 + \frac{\varepsilon}{2}) opt \\ &\leq (1+\varepsilon) opt, \end{aligned}$$

which completes the proof of Lemma 4.4. □

Lemma 4.3 and Lemma 4.4 imply that invariant $\kappa(i)$ holds for each $i \in [720k\varepsilon^{-3}]$. Recall that inequality (6) says that $\kappa(0)$ is true. Putting together, we know that $\kappa(i)$ remains true for each $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$. The following result is called Chernoff's bound [1], which will be used to analyze the client-connection costs.

Lemma 4.5. Given a real number $p \in [0, 1]$ and w independent random variables $x_1, \ldots, x_w$ satisfying $x_i \in \{0, 1\}$ and $\Pr[x_i = 1] \geq p$ for each $i \in [w]$, it is the case that $\Pr[\sum_{i=1}^{w} x_i < (1-\tau)pw] < e^{-\frac{\tau^2 pw}{2}}$ for any $\tau \in (0, 1)$.

We are now ready to prove Lemma 2.1.

Lemma 2.1. Inequality $\sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}) \leq (1+\varepsilon) opt$ holds with constant probability for any constant $\varepsilon \in (0, 1]$.

Proof. Let $x_0, \ldots, x_{720k\varepsilon^{-3}}$ denote a sequence of independent random variables satisfying $x_i \in \{0, 1\}$ and $\Pr[x_i = 1] = \varepsilon^3/360$ for each $i \in \{0, \ldots, 720k\varepsilon^{-3}\}$. Let $p = \varepsilon^3/360$ and $w = 720k\varepsilon^{-3}$. Considering an integer $i \in \{0, \ldots, w\}$, invariant $\kappa(i)$ says that either $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq p = \Pr[x_i = 1]$, or $\sum_{j=1}^{k^*} |C_j^*| d(f_j^*, \mathcal{R}_i) \leq (1+\varepsilon) opt$ and Lemma 2.1 holds. For the case where inequality $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq p$ holds for each $i \in \{0, \ldots, w\}$, we have

$$\begin{aligned} \Pr[\mathbb{H}_{w+1} = \mathbb{C}^*] &\geq \Pr[\sum_{i=0}^{w} x_i \geq k^*] \geq \Pr[\sum_{i=0}^{w} x_i \geq k] \\ &= 1 - \Pr[\sum_{i=0}^{w} x_i < k] = 1 - \Pr[\sum_{i=0}^{w} x_i < \frac{1}{2} pw] \\ &> 1 - e^{-\frac{pw}{8}} = 1 - e^{-\frac{k}{4}} \geq 1 - e^{-\frac{1}{4}}, \end{aligned} \tag{17}$$

where the first step is due to the assumption that $\Pr[|\mathbb{H}_{i+1}| > |\mathbb{H}_i|] \geq p = \Pr[x_i = 1]$ for each $i \in \{0, \ldots, w\}$, the second step is due to the fact that $k^* \leq k$, the fifth step follows from Lemma 4.5, and the last step is due to the fact that k is a positive integer. The definition of $\mathbb{H}_{w+1}$ implies that we have $d(f_j^*, \mathcal{R}_{w+1}) < (1+\varepsilon) opt_j$ for each $j \in [k^*]$ for the case where $\mathbb{H}_{w+1} = \mathbb{C}^*$, which in turn implies that $\sum_{j=1}^{k^*} d(f_j^*, \mathcal{R}_{w+1}) < (1+\varepsilon) opt$. Combining this with inequality (17), we know that Lemma 2.1 is true. □

C Proof of Lemma 3.1

Lemma 3.1. It can be assumed that $d_{\max} \leq O(n^3\varepsilon^{-1} d_{\min})$ and $o_{\max} \leq O(n^2\varepsilon^{-1} o_{\min})$ for any constant $\varepsilon \in (0, 1]$, which incurs a $1+\varepsilon$ multiplicative overhead in the approximation ratio of our algorithm for $\mathcal{I}$.

Proof. We will show how to modify instance $\mathcal{I}$ to make the desired inequalities valid. Define $o^*_{\max} = \max_{f\in\mathcal{S}^*} o(f)$ as the maximum facility-opening cost in the considered optimal solution to $\mathcal{I}$, and define $d^*_{\max} = \max_{c\in C} d(c, \gamma^*(c))$ as the maximum connection cost associated with the clients from C in the solution. Since there are no more than $|\mathcal{F}|$ possible values for the facility-opening costs and $|C||\mathcal{F}|$

possible values for the client-connection costs, the values of $o^*_{\max}$ and $d^*_{\max}$ can be correctly guessed by brute-force enumeration with an $O(n^3)$ multiplicative overhead in the running time of the algorithm. Define $\psi = n(d^*_{\max} + o^*_{\max})$. The definitions of $o^*_{\max}$ and $d^*_{\max}$ implies that $\psi \in [opt, n \cdot opt]$. We modify instance $\mathcal{I}$ based on ψ as follows. For each $x, y \in \mathcal{C} \cup \mathcal{F}$ satisfying $d(x, y) > n\psi$, let $d(x, y) = n\psi$. The fact that $n\psi \geq n \cdot opt$ implies that such two points cannot be connected by any $O(1)$-approximate solution to the instance. For each $x, y \in \mathcal{C} \cup \mathcal{F}$ satisfying $d(x, y) < \frac{1}{2}n^{-2}\varepsilon\psi$, let $d(x, y) = \frac{1}{2}n^{-2}\varepsilon\psi$. For each $f \in \mathcal{F}$ satisfying $o(f) < \frac{1}{2}n^{-2}\varepsilon\psi$, let $o(f) = \frac{1}{2}n^{-2}\varepsilon\psi$. Increasing the minimum distance and facility-opening cost increases the cost of any solution by no more than $n \cdot n^{-2}\varepsilon\psi = n^{-1}\varepsilon\psi$, which can be upper-bounded by $\varepsilon \cdot opt$ due to the fact that $\psi \in [opt, n \cdot opt]$. Let $\mathcal{I}'$ be the resulting instance, and denote by opt' the cost of an optimal solution to $\mathcal{I}'$. Based on the argument above, we have

$$opt' \leq (1 + \varepsilon)opt. \tag{18}$$

Let $d'_{\max}$ and $d'_{\min}$ denote the maximum and minimum distances between the input data points of $\mathcal{I}'$, respectively. Similarly, let $o'_{\max}$ and $o'_{\min}$ be the maximum and minimum facility-opening costs in $\mathcal{I}'$, respectively. It can be verified that $d'_{\max} \leq O(n^3\varepsilon^{-1}d'_{\min})$ and $o'_{\max} \leq O(n^2\varepsilon^{-1}o'_{\min})$. Moreover, it was known that increasing the minimum distance and decreasing the maximum distance between different points cannot invalidate triangle inequality (see Appendix C in [2]). Thus, we know that the distance function of $\mathcal{I}'$ still satisfies triangle inequality since only the maximum and minimum distances between the points are modified when the instance is constructed. Consequently, Lemma 3.1 is true if each λ-approximate solution to $\mathcal{I}'$ is also a $\lambda(1+\varepsilon)$-approximate one to $\mathcal{I}$ for each constant $\lambda > 1$.

It remains to consider the $O(1)$-approximate solutions to $\mathcal{I}'$. Given a constant $\lambda > 1$ and a λ-approximate solution to $\mathcal{I}'$, we know that its cost for $\mathcal{I}$ can be upper-bounded by its cost for $\mathcal{I}'$ (due to the fact that none of the truncated edges is used by the solution, as discussed above), and is no more than $\lambda \cdot opt' \leq \lambda(1+\varepsilon)opt$ due to inequality (18). Thus, Lemma 3.1 is true. □

D Proof of Lemma 3.2

Lemma 3.2. Let f'_i denote an arbitrary facility from $\{f \in Q_i : \ell(f) = i\}$ for each $i \in [k^*]$, then we have $\sum_{i=1}^{k^*}(o(f'_i) + d(C^*_i, f'_i)) \leq (1 + \varepsilon)\sum_{i=1}^{k^*} o(f^*_i) + (3 + O(\varepsilon))opt$ for any constant $\varepsilon \in (0, 1]$.

Proof. The members of Q_i are illustrated in Figure 1. We have

$$\begin{aligned} o(f'_i) + d(C^*_i, f'_i) &\leq o(f'_i) + d(C^*_i, g_i) + |C^*_i|d(g_i, f'_i) \\ &\leq o(f'_i) + opt_i + |C^*_i|d(g_i, f^*_i) + |C^*_i|d(g_i, f'_i) \\ &\leq o(f'_i) + opt_i + (2+\varepsilon)|C^*_i|d(g_i, f^*_i) \\ &\leq (1+\varepsilon)o(f^*_i) + opt_i + (2+\varepsilon)|C^*_i|d(g_i, f^*_i) \end{aligned}$$

for each $i \in [k^*]$, where the first two steps are due to triangle inequality, the third step is due to the fact that $f'_i \in Q^1_i \subseteq Q_i$ and the definition of Q^1_i, and the last step follows from the fact that $f'_i \in Q^2_i \subseteq Q_i$ and the definition of Q^2_i. Summing both sides of the inequality over $i \in [k^*]$, we get

$$\begin{aligned} &\sum_{i=1}^{k^*}(o(f'_i) + d(C^*_i, f'_i)) \\ &\leq \sum_{i=1}^{k^*}((1+\varepsilon)o(f^*_i) + opt_i) + \sum_{i=1}^{k^*}(2+\varepsilon)|C^*_i|d(g_i, f^*_i)) \\ &\leq \sum_{i=1}^{k^*}((1+\varepsilon)o(f^*_i) + opt_i) + (2 + O(\varepsilon))opt \\ &= (1+\varepsilon)\sum_{i=1}^{k^*} o(f^*_i) + (3 + O(\varepsilon))opt, \end{aligned}$$

where the second step is due to Lemma 2.1. □

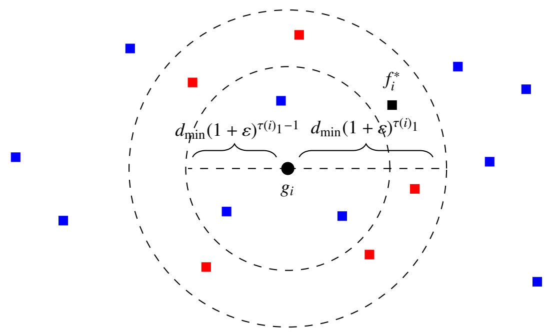

Fig. 1: The square points are the facilities from $\mathcal{F}$ whose opening costs lie in $[o_{\min}(1+\varepsilon)^{\tau(i)_2-1}, o_{\min}(1+\varepsilon)^{\tau(i)_2}]$, where the red ones are the members of $Q_i \backslash \{f^*_i\}$.

References

1. Zhou Q M, Calvert A, Young M. Singletons for simpletons revisiting windowed backoff with Chernoff bounds. Theoretical Computer Science, 2022, 909: 39-53
2. Ahmadian J, Norouzi-Fard A, Svensson O, Ward J. Better guarantees for *k*-means and Euclidean *k*-median by primal-dual algorithms. SIAM Journal on Computing, 2020, FOCS17: 97-156