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Modern interpretations of probability

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Abstract

Probability, along with logic and fuzziness, has become an essential framework for studying and managing complex, uncertain, and dynamic processes across science, engineering, and society, particularly in an era marked by globalization, incomplete information, and rapid technological change. The present work continues and further develops the exploration of the category of "probability" in its application to the technical field, particularly in radio engineering and electronics, as previously explored by the authors. In this interpretation, the concept of "probability" is examined across various areas of human activity, rendering it relevant to a broad range of specialists engaged with random processes in their respective fields. This study analyzes the varying interpretations underlying the theoretical foundations and practical applications of probability, logic, and fuzzy systems. This variability arises from differing understandings of the philosophical meaning of probability and logic, the objective principles governing the interconnection of probabilistic and logical reasoning, and the formation of fuzzy set theories and associated engineering tools. These tools have become increasingly useful in implementing the life cycle of structurally complex long-range technical systems, particularly in fields such as energy, telecommunications, military, and space technology. Moreover, the theory of fuzzy multitudes and related engineering approaches has gained relevance in studying climate change and the environmental ecology surrounding humans. They help establish the current state and predict future trends in social systems, wildlife populations, and other complex, dynamic environments.

Keywords: Probability; Logic; Multitude; Fuzziness; Incomplete information; Engineering activities

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1. Introduction

Probability, as a mathematical category, is a numerical characteristic of the degree of possibility of a random event occurring under certain specified conditions that can be repeated an unlimited number of times. Probability theory is a branch of mathematics that focuses on determining the probabilities of events related to given random events. Based on the given probabilities of individual random events, the probabilities of other events related in any way to the primary events can be determined. This theory also studies random variables and random processes. The mathematical apparatus of this theory is adapted to the study of mass phenomena in society, science, and technology. The methods of probability theory play an important role in the processing of statistical data.

Several derived probability concepts are widely used, including classical, a priori, a posteriori, conditional, statistical, geometric, and subjective probability. In technology, probability categories such as error probability, deviation probability, recovery probability, confidence probability, unconditional, relative, transitional, and predicted are common.

The globalization processes of the modern era extend beyond economic relations between countries and influence many other critical aspects of human life, determining, among other things, the very conditions of human existence. These processes affect areas such as climatology, the expansion of migration flows, trends in regional and global overpopulations, the depletion of food and energy resources, and more.

The modern stage of development in science, engineering, and technology is deeply influenced by globalization processes, marked by rapid, sometimes explosive, advancements. These shifts have led to significant, often revolutionary, transformations in scientific research, engineering, pedagogy, psychology, sociology, and other disciplines. A defining feature of these changes is their scale, massiveness, multidirectionality, and simultaneous coverage of objects of both animate and inanimate nature. Despite occasionally being logically structured, the outcomes of such large-scale processes are inherently difficult to predict with certainty. This is because any information about their progression (which today is becoming increasingly important) under any conditions will be characterized as incomplete and inaccurate, and presented in both numerical and non-numerical forms.

Modern research and forecasting of these processes are largely associated with three categories: *probability, logic, and fuzziness*. These have become integral across various professional fields, including engineering, sociology, medicine, and psychology. Specialists investigating large systems and global processes inherent in animate and inanimate nature often face the challenge of making appropriate decisions under conditions of uncertainty, characterized by incomplete, inaccurate, non-deterministic, often non-numeric information.

In such conditions, applying well-developed classical methods of the theory of functions and functional analysis to study the corresponding objects becomes limited, inefficient, and, quite often, impossible. At the same time, the concepts of probability, logic, and fuzziness offer versatile research tools in various fields of science and technology, though each comes with multiple interpretations. These approaches prove effective and lead to success only under certain conditions. At the same time, a specialist's ability to understand and appropriately select from these tools, to deeply comprehend their underlying principles, enables a more conscious and reflective approach toward their field of study and cognitive processes. This self-awareness empowers professionals to manage these processes, directing them to achieve the desired result.

This study analyzes the varying definitions and interpretations of the concepts of probability, logic, and fuzziness. It identifies the commonalities and distinctions inherent in these concepts and provides a rationale for the inevitability and objectivity of their interpenetration, considering their potential application.

Numerous definitions and interpretations of the concept of probability have been developed to date. While not all of them are equally applied in theoretical research and practical activities of specialists from different fields of knowledge, both technical and humanitarian, each interpretation has merit within specific contexts. Their legitimacy is supported by years of successful application. In some respects, a similar situation has emerged in the field of logic, which, since its inception, has evolved to encompass significantly broader features. These developments have expanded the theoretical foundation of logical reasoning and the scope of usage of this scientific direction.

The development of society, science, engineering, and technology has increasingly demanded clear answers to critical questions such as:

(i) How can we overcome or, at least, evaluate the incompleteness and inaccuracy of information regarding the events under study, especially non-mass, non-systematic, and single events?
(ii) Is there a connection between the categories of probability and logic, and how broad and deep is that connection?

Pursuing answers to these questions has become particularly relevant in studying social behavior, designing large telecommunication and energy systems, developing weapon and defense systems, and building expert and artificial intelligence systems. These challenges have fueled further investigation into overcoming internal contradictions and limitations in probability theory and logic. This has led to a mutual penetration, and in some cases, a merging (if we can apply such a metaphor), of probability and logic. This has given rise to other useful theoretical and practical constructs, most notably fuzzy theory (fuzzy set theory). Today, fuzzy theory is considered (and rightfully so) the most adequate and effective tool for solving many practical problems in various fields of human activity.

2. Interpretations of the concept of probability

Among the considered concepts (probability, logic, and fuzziness), the concept of probability has undergone the most significant variation of definitions and interpretations throughout its existence. Within this context, each proposed interpretation of probability always corresponded to certain conditions and goals regarding the practical application of probabilistic tools or their theoretical justification. Nevertheless, scientific discourses around the concept of probability are ongoing, which may indicate the possibility (or rather, the necessity) of the emergence of new interpretations of this concept and new areas of usage of this field of knowledge.

In the 20th century, the concept of probability and the tools based on this concept became the focus of attention not only for specialists in technical fields, where the randomness of the realization of events began to gain more and more weight but (as a consequence) also for the philosophical community. After all, immersion into the content and essence of probability inevitably leads to philosophical reflections and conclusions. The spread of the application of probabilistic tools as the basis of many algorithms for constructing new systems in telecommunication, energy, military, and space technology, as well as in comprehensive studies of human society, has led to the need to comprehend the very concept of probability. This has prompted scholars in the field of logic and philosophy, which are traditionally associated with the study of the essence of scientific theories, to examine the internal content and foundational meaning of probability.

Probability theory has become an independent branch of mathematics in the last century and is finding new application areas. However, as the application of this theory becomes wider and more diverse, the interpretation of probability and the content of probabilistic conclusions become more difficult. Consequently, it becomes more difficult to find an (better would be a universal, unified, generalized) interpretation of probability, one that not only satisfies the logical and methodological demands of relevant research but also aligns naturally with everyday intuitions and commonly accepted understandings of probability based on its established applications. The existence of a mathematical theory of probability, unfortunately, does not (as will be demonstrated later) provide grounds for forming a unified view of the content of probabilistic judgments and conclusions across the diverse contexts in which they are applied.

If we consider probability as a philosophical category that characterizes the reality surrounding a person, then we should state its bifurcation, the separation of objective probability and subjective probability.

Objective probability is a concept that characterizes a quantitative measure of the possibility of the occurrence (realization) of an event under certain conditions. This type of probability characterizes the objective properties of mass events of a random nature and is studied by the *mathematical theory of probability.* Mathematical probability is an objective quantitative characteristic of the degree (measure) of the possibility of the realization of a certain event, which can be repeated an unlimited number of times under specific predetermined conditions.

Subjective *probability* allows us to analyze the peculiarities of a person's subjective cognitive activity under conditions of uncertainty (incomplete certainty). In this case, probability acts as a measure of subjective confidence in realizing a certain event. This confidence is determined, firstly, by the information available (or absent) in a given person regarding the event under study, and secondly, by the psychological features of this person. This psychological aspect plays an important role in how individuals assess the likelihood of a given event's occurrence. The studies of subjective probability, as shown below, played a significant role in developing scientific directions, where probabilistic and logical characteristics of observable events are combined, as well as in the origin and development of the theory of fuzzy sets.

3. Classical (symmetric) interpretation of probability

The classical interpretation of probability was historically the first attempt to estimate the results of the realization of events.[1,2] The initial probabilistic representations, concepts, and methods emerged from analyzing situations that manifest themselves in gambling. It is important to note that gambling is usually organized in a manner that makes different realizations of the corresponding event *equally probable.* In the example of throwing a dice (which has become classical), the events connected with the falling of any face of this dice are equally possible. From this condition, it is easy to calculate the probability of events concerning falling out of one of the faces, without the necessity to address the experiment. If the die is made carefully (and it is the obligatory condition of identicalness of events), the probability of falling out of any face will be 1/6. The definition of probability using this approach is detailed by Jacob Bernoulli in his work *The Art of*

Conjecturing.[3] The cover of the English-language edition of the book *The Art of Conjecturing* is shown in Figure 1.[4]

Significant attention was devoted to the concept of probability by Pierre-Simon Laplace, who, in his work *A Philosophical Essay on Probabilities*,[5] most fully and comprehensively defined the classical interpretation of the concept of probability as the *ratio of the number of favorable realizations of some event to the number of all possible realizations of this event, and different realizations of the event under study should be equally possible (symmetric, equally, or possible).* Portrait of Pierre-Simon Laplace is shown in Figure 2.[6]

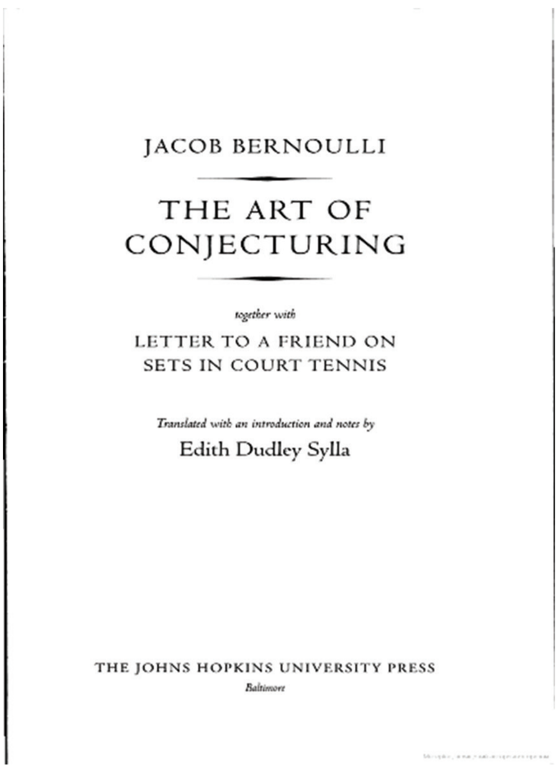
Thus, according to the classical interpretation of probability, its quantitative determination does not involve any empirical study. The analysis of the possibility of realization of the corresponding event provides, in *advance, the* determination of the probability value that corresponds to such realization (it is due to this feature that the classical interpretation of probability is often called *a priori*). The classical interpretation of the concept of probability, based on the assumption of the *equal possibility of separate realizations of the events under study,* contains two logical contradictions.

First, it is very difficult, or, more precisely, practically impossible, to guarantee and to check in practice the same possibility of realization of the events under study. The concept of the same possibility of realization of events should be attributed to the realm of the ideal, covering the entire research space (observation), and considered as an idealized model of the corresponding process.

Second, there is the existence of a false circle regarding the definition of probability itself. In fact, within the framework of classical interpretation, the probability is defined through the same possibility of realization of events, which essentially means the same probability (equal probability) of realization of these events. At the same time, the introduction of the concept of equal probability, in turn, supposes the existence (preliminary establishment) of the definition of probability.

To overcome the difficulties of the classical interpretation of probabilities, it was proposed to accept as true those judgments and conclusions for which there are no sufficient grounds for rejection, a notion that was formulated as the *principle of indifference. By* this principle, the probability of the realization of two events is proposed to be considered the same (regardless of whether this provision is fulfilled in reality), which provided that there are no grounds for the opposite assumption.

A detailed analysis of the principle of indifference (which can be regarded as the principle of the plausibility of conclusions regarding the properties of the investigated events) and its usage both in theoretical studies and in practical activity indicates the existence of two directions of the classical concept of probability. The first direction recognizes the identical probability of realization of all investigated events of a given population *under any conditions,* and the second only in the *presence of an additional condition,* namely, the condition of fulfillment of the principle of indifference. In the first of these



JACOB BERNOULLI

THE ART OF CONJECTURING

together with

LETTER TO A FRIEND ON SETS IN COURT TENNIS

Translated with an introduction and notes by

Edith Dudley Sylla

THE JOHNS HOPKINS UNIVERSITY PRESS

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Figure 1. Cover of the English-language edition of the book *The Art of Conjecturing*[4]

Figure 2. Portrait of Pierre-Simon Laplace by Johann Ernst Heinsius (1775). Image obtained from the Wikimedia Commons website.[6]

directions, the condition of equal probability is considered (by default) as an attribute; that is, it is treated as an inherent attribute of the events within the investigated population. In contrast, in the second one, the probability is “tied” to the available knowledge (available information) about the events of this population. Consequently, the first model should be considered idealized and objective, while the second is subjective (epistemological). This distinction is essential; thus, not accounting for it can lead to erroneous results. The principle of indifference assumes symmetrical consequences of the events under study, that is, the same probability of their realization. However, in contrast to the “primary” concept of classical probability, it is not accepted as itself but is established based on analysis (logical, physical, and chemical), which is based on the available knowledge about the totality of events under study. Despite the external difference between the two models of the classical interpretation of probability, they both rely on a common basis – symmetry (equal probability) of the events of the investigated totality. Only in the first model is symmetry considered an attribute of the events. In contrast, in the second model, it is confirmed by the available (at the moment) knowledge about the investigated set of events. The latter gives grounds to consider the probability subjective, which is determined using the principle of indifference.

Thus, the classical interpretation of the concept of probability contains internal contradictions. In general, this interpretation cannot be taken as a definition of the concept of probability, because it points mainly (if not exclusively) to a method of calculating probability and one that “works” only in particular (simplest or specially created) situations. At the same time, this interpretation played an appropriate constructive role in the history of mathematics because it started and properly directed the search for answers to the question of the calculation of probability in general conditions, and this led to the formation of a new mathematical theory, which was later named probability theory.

4. Frequency (statistical) interpretation of probability

The classical interpretation of probability is considered to be related to the analysis of the realization of an act or an event, indicating the intentions of the possibility of such realization. However, in practical activity, most of us have to deal with sets (classes) of events, which actualize the task of studying the properties of such sets, leading to a different (frequency) interpretation of probability.

It is evident (at least there is no substantial reason to deny it) that the measure of the likelihood of an event’s occurrence depends on how frequently that event is observed within the context of a broader set of related events. The measure of *the possible realization of an event depends on the frequency of its realization.* The more often the event is repeated, the higher the measure of its possible realization. Such events were later called *mass random events,* because they, firstly, differ from regular events, which appear regularly. Second, they are not unique, single events, where the degree of possibility of their realization is senseless and cannot be connected with the frequency of their realization. On this basis, the *interpretation of probability as a relative frequency of realization of a mass random event,* namely, the frequency (statistical) interpretation of probability, emerged and began to develop.

The frequency (statistical) interpretation of probability has its roots in ancient science. For many centuries, this idea and the corresponding conclusions associated with it have been used at an intuitive level, applying it in various fields of human activity. However, the English scientist John Venn first conducted consistent and comprehensive studies of this concept and presented them in *The Logic of Chance*, published in 1866. Since then, the frequency interpretation of probability has been gaining popularity among specialists whose activities are related to processing large arrays of repetitive data (i.e., statisticians). This reaction from statisticians is quite understandable because most statistical problems cannot be reduced to a set of events, the probability of realization of which is the same. Consequently, it is impossible to apply the classical interpretation of probability.

According to the frequency interpretation, the probability is determined through the relative frequency of events directly or indirectly, based on observations of the corresponding process or the basis of specially designed experiments and tests.[4] J. Venn defined probability as a *limit value of the relative frequency of event realization at a large number of tests.* At the same time, the *“large number” concept was not specified. It* was proposed to take *relative frequency* as the primary concept when defining probability. Since the relative frequency is determined by the practical implementation of the corresponding procedure, this probability is called *empirical probability.*[7]

The frequency interpretation of probability was further developed in the 20th century in the works of Richard Mises (Figure 3[7]), who significantly supplemented and expanded Venn’s conclusions. Mises considered probability as a mathematical artifact reflecting not finite sequences of observational results, which have corresponding properties, that is, first, an artifact was introduced, and then a comparison of this artifact with reality was carried

Figure 3. Richard Edler von Mises. Image obtained from the Wikidata website.[7]

out.[8] This artifact was based on the *relative frequency of the* realization of a mass random event with sufficiently long observations. At the same time, the probability value in all such situations was determined based on statistical procedures and was called *statistical probability. Given* that statistical probability is numerically determined by relative frequency, it is often called *frequency probability.* This approach is adopted in statistics today, where probability is identified with the relative frequency of the realization of a mass random event *with sufficiently long observations.* Note that in the definition of this interpretation of probability by Mises (and Venn), the duration of observations is not predetermined. Instead, it is proposed to be established during the research process, that is, simultaneously with the observation of the analyzed event and based on the data collected through these observations. In addition, the conditions for realizing events must remain unchanged to obtain an adequate result throughout the observation process.

Statistical probability is the limit value of the sequence of relative frequencies of the realization of the expected event among all observed and analyzed events. However, this limit remains theoretically unproven and does not allow us to accept Mises's constructions as an appropriate theory.

Long-term practice has established that for many mass phenomena (for streams of events), the relative frequency tends to stabilize as the number of observations increases. Such stability of frequencies of mass phenomena is an *objective regularity,* independent of human will and desire. The practice also shows that if it is possible to calculate the probability of events based on the property of symmetry of these events, then the relative frequency is quite an acceptable estimate of the specified probability.

At the same time, in the general case, when the exact value of probability based on the property of symmetry of events cannot be calculated, an *inductive* assumption is made that the relative frequency of such mass, recurrent events is very close to some number, which is called probability. Consequently, inductive inference is necessary to support the frequency interpretation of probabilities. Still, given that any inductive inference is a departure from the boundaries of the known, it inevitably leads to difficulties of a theoretical nature.

An essential (almost the most important) problem arising in applying methods based on the frequency interpretation of probability is the verification of the correctness of the obtained results. This is because verifying the truth (adequacy) of the obtained probabilistic conclusion is quite natural in the study of any events occurring both in animate and inanimate nature. In general, neither the verification nor the falsification procedure can verify this.

If we consider classes of events as sets of finite power, then probabilistic inference would directly indicate the arithmetic ratio of the powers of these classes. In this case, the probabilistic inference could be checked by presenting all elements of both classes, followed by a simple count. If we consider infinite classes of events or an infinite number of finite statistical sequences, then no finite number of observations can definitively confirm or definitively reject the probabilistic conclusion made. One cannot (at least theoretically) exclude that a given finite sequence of observations is a certain fluctuation, a significant deviation of the relative frequency in a given sequence from the relative frequency in the entire infinite class of analyzed events.

In this context, it should be noted that the common practice among specialists from different fields of discarding individual observations (experiments) or experimental results that differ significantly from the results of the bulk of observations realized within the framework of this study, regardless of circumstances, cannot be recognized as methodologically flawless. Such an exclusion procedure should be treated with caution, applying it only after a thorough analysis of the relevant situation and establishing the reasons, if possible, for this difference.

An important problem related to frequency probability is the logic of constructing observations. This problem highlights the necessity of a clear formulation of the conditions that sequences of events must satisfy for the frequency interpretation of probability to be validly applied. This is because the calculated frequency-based probability often depends significantly on the sequence of realization of events of a particular population (on the organizational basis of observations). This leads to the philosophical conclusion that to obtain an adequate result, the frequency interpretation of probability should be applied not to the fact of realization of events of the studied population, but rather to the organization of the process of realization of these events, that is, to the process of research and the situation "accompanying" this research.

Here, recalling the dispositional interpretation of probability proposed by the Austrian–British philosopher and sociologist Karl Raimund Popper is appropriate. According to this interpretation, probability is not a property of events in the investigated population but determines the *dispositional properties of* some (preliminarily accepted) investigated situation.

Unlike the statistical one, the dispositional interpretation of probability gives probability an ontological status because probability is an inherent property of this or that objective situation (disposition), which can manifest itself in various forms regardless of our knowledge. Popper's conclusions have not found wide support in the scientific community (mainly about the possibility of applying the concept of frequency probability to assess the likelihood of the realization of single events). However, they should be considered when preparing studies, the results of which are supposed to be processed based on the concept of frequency probability.

Thus, the frequency probability should be considered as a mathematical artifact characterizing quantitative regularities (ratio) of two sets of events and cannot be applied when we are talking about non-repeating, rarely repeated, or single events. In practice, it is usually impossible to fulfill the conditions that would give confidence in the existence of a limit value of the relative frequency of the realization of events corresponding to the frequency probability. In particular, in observing events, it is practically impossible to ensure the consistency of the conditions in which these observations are carried out. This always gives certain reasons to doubt that the obtained value of the relative frequency of realization of events, even with its relative stabilization within the framework of the observations, will adequately reflect the probability of realization of the analyzed event when the number of observations rises to infinity.

The existence and practical necessity of studying events that can be observed exclusively in constantly changing conditions lead to the concept of *hyper-random* events.[9-12] A characteristic feature of such events is the impossibility of applying the described idea of frequency probability because any increase in the observed number of event realizations will not limit the frequency of event realization under the specified conditions. At the same time, numerous applied problems involve the study of hyper-random events, providing a strong rationale for viewing the development of this branch of probability theory, particularly within its frequency interpretation, as a promising direction.

5. Axiomatic interpretation of probability

The axiomatic approach to constructing probability theory is a turning point in developing this branch of mathematical science. The introduction of axiomatics, as the basis of all further conclusions and provisions, turns probability calculus into a rigorous mathematical discipline, free from implicit assumptions and logical contradictions. However, the flawlessness of the theory was achieved at the expense of a *complete abandonment of any substantive laziness concerning the concept of* probability *itself*. Within the axiomatic theory, probability as a concept is not defined or, more precisely, is defined formally as some appropriately specified function, a mathematical artifact satisfying an established system of axioms. In this case, the question is not even posed as "what is the probability of a real event?", and how to calculate or measure this probability in the sphere of practical activity is not considered.

The axiomatic interpretation (in this case, the word "theory" is also quite applicable) of probabilities is related to an idealized mathematical reality. Among the existing axiomatic theories of probabilities, the most general and transparent one is considered to be the theory proposed by Andrej Nikolayevich Kolmogorov in 1929. Kolmogorov's axiomatic probability theory[13,14] is considered in mathematics as a separate direction of measure theory, relying on the concepts and methods of set theory. Within the framework of this axiomatic theory, the very concept of probability is considered an undefined initial *concept. The* motives for constructing an axiomatic theory of probability and his attitude to such a theory with A.N. Kolmogorov are presented in the following words:

> Probability theory as a mathematical discipline can and should be axiomatized completely in the same sense as geometry and algebra. This means that once the names of the objects to be analyzed and their basic relations have been given, and the axioms to which these relations must satisfy have been introduced, all further teaching of probability theory must be based solely on the axioms introduced, without relying on the ordinary concrete meaning of the objects and their relations indicated.[13(p.1)]

The portrait of Andrej Nikolayevich Kolmogorov is shown in Figure 4.[15]

In the framework of Kolmogorov axiomatics, the probability is defined as a numerical function *P(A) of an*

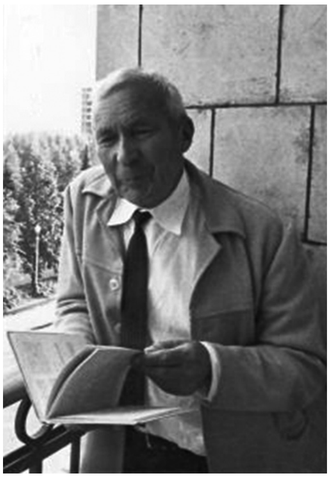

Figure 4. Andrej Nikolayevich Kolmogorov. Image taken from Wikipedia.[15]

event A *on* the set of all events determined by a series of corresponding observations. It is assumed that the specified set contains not only all events but also all possible combinations of these events, and the introduced function satisfies the conditions $0 \leq P(A) \leq 1$, and $P(A) = 1$ if A *is* a valid event. Besides, the truth of the relation $P(A \vee B) = P(A) + P(B)$ is accepted, which makes the simultaneous realization of events A and B impossible. *What is* important in the above axiomatics is that it is *considered on the set of events whose measure equals one.*

This interpretation of probability describes the formal properties of this concept, built on mathematical artifacts, operates with such artifacts, and, based on them, leads to certain conclusions. Such a theory says nothing about the nature of the phenomena to which it is applied and which it describes. In this interpretation, the probability is a function taking values in the interval from zero to one, and satisfying appropriate conditions, as the events themselves must fulfill appropriate conditions. All these conditions and properties are introduced to achieve the appropriate logical structure of the theory, and its internal incontestability, aimed at "servicing" the theory. In practice, the events and the specified function may not correspond (and often do not correspond) to the established conditions. Notably, the abstract concept of probabilities represents certain properties of frequencies observed within the framework of the corresponding observations, but they are not completely reduced to these frequencies. At the same time, the mathematical apparatus of probability theory (in its axiomatic interpretation) is widely and fruitfully used in the practical activities of specialists in various technical and humanitarian knowledge fields.

The widely developed modern mathematical interpretation of axiomatic probability provides standardized definitions of probability and its meanings. Probability is a number $P(A) \in [0,1]$ characterizing the degree of possibility of occurrence of the determining event A. For example, $P(X = a)$ means the probability that the random variable X takes the value a.

Classical probability is the ratio of the number of cases favorable to a given event to the total number of equally likely cases. The prior probability of event B_i is the probability of event $P(B_i)$ before the experiment. Conditional probability is the probability of event A, given that some other event Bi, $P(A/B)$, is carried out at the same time. The posterior probability is the probability of event B_i after an experiment in which event A occurred, $P(B_i/A)$.

Bayes' theorem relates the categories of prior conditional and posterior probabilities: if B_1, B_2, B_n is a complete system of events, $P(B_k)$ is the prior probability of events B_k, and $P(A/B_k)$ is the conditional probability of event A given that event B_k has occurred, then the posterior probability $P(B_k/A)$ of event B_k given that event A has occurred is defined in Equation I.

$$P\left(B_k \;/\; \mathrm{A}\right) = \frac{P\left(B_k\right) \cdot P(A \;/ B_k)}{\sum_{j=1}^{n} P(B_j) \cdot (A \;/ B_j} \qquad \text{(I)}$$

This expression reflects the so-called Bayesian approach to statistical problems.

The difference between the Bayesian approach and other statistical approaches is that even before the data are obtained, the researcher considers their degree of confidence in possible models and represents it in the form of prior probabilities.

Geometric probability is the probability of hitting a certain part of the area on which a uniform distribution is defined. The statistical probability is the relative frequency of events in a sufficiently long series of experiments. Subjective probability is the probability of an event according to the subjective opinion of an expert. In the axiomatic interpretation of probability, the key is the definition of the concept of random events, which are formalized using a probability space that is given by the triad (Ω, Ψ, and P), where Ω is the space of elementary events $\omega \in \Omega$, Ψ is the σ-algebra of subsets of events, which is defined as the algebra of sets S containing the union of any number of its elements (or Borel field), and P is the probability measure (probability) of a subset of events.

The accurate use of probability theory is provided by accepting the physical hypothesis of ideal statistical stability (stability) of parameters and characteristics of physical phenomena. However, experimental studies on large observation intervals of various processes of different physical natures show that the hypothesis of ideal statistical stability is not confirmed. Statistical estimates formed on relatively small intervals of observations have relative stability, that is, with an increasing volume of statistical data, estimate value fluctuations decrease. However, starting from some critical volume, with increasing quantitative data, the fluctuations practically do not change, and sometimes increase. The violation of statistical stability in the real world means that the concept of probability has no physical interpretation.

Probability is a mathematical abstraction. The causes of violation of statistical stability can originate from various processes, for example, the inflow into an open system from external "substance," energy, and information, which feed unsteady processes.

The study of violations of statistical stability and the search for effective means of an adequate description of real phenomena of the surrounding world, taking into account these violations, led to the creation of a new physical and mathematical theory of hyper-random phenomena by the Ukrainian scientist Igor Illich Gorban.[9-11]

It is possible to describe hyper-random phenomena using the triad (Ω, Ψ, $\ddot{\Upsilon}$, and P). Ω is the space of elementary phenomena, $\omega \in \Omega$, Ψ is the Borel field, $\ddot{\Upsilon}$ is the set of conditions $g \in \ddot{\Upsilon}$, and P_g is a probability measure of subsets of events that depend on conditions g.

A hyper-random event A can be interpreted as an event whose frequency of occurrence $P_N(A)$ does not stabilize with increasing number of experiments N and has no boundary (limit) at $N \to \infty$. In this case, the frequency of events does not have statistical stability. For example, if a random variable X is completely characterized by a probability distribution $F(x)$, then a hyper-random variable $X = [X/g \in \ddot{\Upsilon}]$ is characterized by a set of conditional distribution functions $F(x/g)$, $g \in \ddot{\Upsilon}$.

The isolation of a new class of random processes, hyper-random variables, allows us to expand the list of metrics for practical use significantly and bridges the gap from the formal-mathematical axiomatic interpretation of the concept of probability to the construction of a new physical and mathematical theory of hyper-random phenomena. According to Gorban, this theory is based on a more general "statistical instability" phenomenon than the alternative "statistical stability" phenomenon, which may be only a partial variant of the former.

The phenomenon of the hyper-random hypothesis and its feature – limited statistical stability – creates the basis for accepting the concept of the world structure on hyper-random principles. The mentioned limitation emphasizes the emergent nature of the key concept of probability, giving it the role of a measure of our ignorance regarding a phenomenon, event, process, and case at a given moment of time of their manifestation.

Events, phenomena, and the world itself are hyper-random. The category "hyper-randomness" parameterizes the "past" as a forerunner, a precursor of the future, but with a share of hyper-randomness. The past and the future are linked by the phenomenon of "probability," or, more precisely, "hyper-probability." It is these categories that, in a general sense, determine predictions for the future in any sphere of science or human existence. If we know the past reliably, with what degree of probability (hyper-probability) is it possible to predict the future?

By the concept of "past," we understand a continuum of events, each of which had its past, present, and future, hence the continuum character of the category "past" itself, its discreteness, and, at the same time, continuity. Each event of this continuum, probable for its past, is in some regular and probabilistic way transformed into the present. We should understand it as an animated (or animated) concrete realization out of a set of unrealized possible variants about which nothing will ever be known in the present or, even more so, in the future.

Thus, each event, the phenomenon of modernity, is the result of choice from a real set of their possible variants in the past, and each of the latter, in turn, is "chosen" from another subset of variants of the even deeper past, and so "infinitely" back in time. In other words, each phenomenon, the event of modernity, is preceded by an "infinite" series of its variants that did not come true in the past. It is "it," one specific one, chosen by-laws, cases, and fate unknown to us, that has reached us.

Hence, the phenomenon of hyper-probability, underlying the existence of the world and its attributes, explains the fact of our unsatisfactory forecasts, which predicted future under stochastic causes of its occurrence (and in separate cases of phenomena, events, and processes, including their individual properties, qualities, characteristics, and parameters). That is, we do not know all the reasons for the event.

After all, when we flip a coin, we can predict with an average probability of 50% the appearance of "heads" or "tails." At the same time, we, living in the 21st century, cannot analytically describe the motion of the hand tossing the coin, taking into account force, speed, mass, distance, acceleration, air resistance, and many other things, with sufficient accuracy, which would determine not probable, but deterministic behavior of the coin. Of course, there are reasons for such behavior of the coin, but they are not known to us now.

This means that in the mathematical function "probability," the argument "time" should be taken into account as well, for after some time, in the future, as the amount of knowledge increases, some stochastic phenomena will pass for a future person into the category of deterministic ones. However, when analyzing complex phenomena, such as social processes, which influence not just the actions of an individual but the behavior of millions of people over decades or even centuries, it is certainly impossible to describe these processes analytically, neither now nor even in the future. Thus, the distance between the probable and the obligatory deterministic is represented by the phenomenon's cognition level. Hence, the corollary follows: the more comprehensive our knowledge, the less likely phenomena appear to us. However, a person remains such that achieving goals causes the emergence of new, more complex goals, demonstrating the restless and boundless curiosity of humans.

Of course, the world is cognizable. But at any moment, the ratio between the cognizable and the cognizable will, unfortunately (or, fortunately), favor the latter. This means that the ratio between deterministic and random phenomena will always be outweighed in favor of the latter, which, in turn, will yield to hyper-random phenomena, processes, and events. Thus, mathematicians, physicists, sociologists, and scientists will always have a job in this world, regardless of its perfection.

6. The interpenetration of the concepts of probability and logic

The interpretation of probability as the degree of possibility of realization of a corresponding event, or the degree of confirmation or non-confirmation of this or that conclusion, has gained relevance not only among mathematicians and specialists who regularly face the need to predict events or process large arrays of data but also in a variety of other fields. Probability as a category of the cognitive process was (and still is) of considerable interest to the philosophical community. This was especially evident at the turn of the 19th and 20th centuries against the background of advances in physics, as well as against the background of the search for new non-binary logical structures. Interpretations of probability as a certain category, and philosophical studies of this category, taking into account multivalued (non-binary) logical structures, led to the emergence of integrated logical-probabilistic structures widely applied in various fields today (Figure 5). The structure of the "probability" category and its development concerning the "logic" category is shown in Figure 5.

The formation of the foundations of mathematical logic as a development of the corresponding philosophical categories, as well as the essence of the mutual penetration (certain fusion) of the concepts of logic and probability, and the consequences of this process, can be understood exclusively within the framework of the corresponding historical discourse.

Logic is believed to have originated in the fourth century BC in ancient Greece and was associated with oratory. At that time, logic and rhetoric reflected mastering the basics of argumentation, logical rules of proof, and forming forms of thinking. It took a long time before the works of Francis Bacon, René Descartes, and Gottfried Leibniz directed logic (already as a science) to form conclusions based on judgments. The English scientist F. Bacon, the founder of scientific induction and inductive logic, was one of the first to turn to experiments in his research. He proclaimed induction, the *movement of thoughts from individual facts (hypotheses) to generalizing conclusions, as a* universal research method. French philosopher R. Descartes supported deductive logic, which *aimed at establishing the truth of specific facts based on general conclusions.* German philosopher G. Leibniz was the first to suggest the necessity of creating symbolic *or mathematical logic.* The portrait of Gottfried Wilhelm von Leibniz is shown in Figure 6.[16]

Moreover, G. Leibniz believed that in the future, it was symbolic logic that has all chances to turn into the "art of calculations," according to which the results of appropriate calculations can solve all contradictions between people: "The only way to improve our judgments is to make them visual, so that errors can be found with the eyes, and if there is a dispute among people, it is necessary to say, "Let's calculate!," then without much formality, it will be possible to see who is right and who is wrong. However, Leibniz's idea about the possibility, necessity, and productivity of reducing judgments to appropriate calculations for many years did not find a response among philosophers or mathematicians. Further development of science showed that not all processes of human cognition of the surrounding world or judgments can be written down in symbolic language. However, Leibniz's idea of introducing appropriate symbolism in logic turned out to be, as reality shows, both effective and productive. Symbolic logic began to be created only in the middle of the 19th century by George Boole, Augustus De Morgan, Charles Peirce, and others. Applying symbolic logic methods to solve problems arising in formal logic allowed not only the solving of these problems but also the formulation and, to a large extent, the solving of problems that could not even be posed within the framework of formal logic. This, of course, caused revolutionary shifts in the early 20th century, both in formal logic and in all scientific directions, in one way or another, related to it.

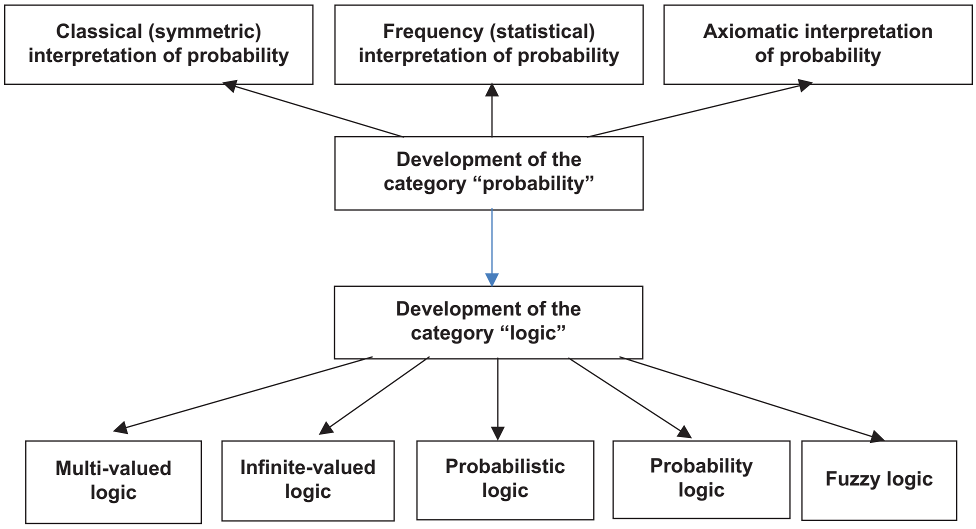

Figure 5. The structure of the probability category and its development concerning the logic category

Figure 6. Portrait of Gottfried Wilhelm von Leibniz by Andreas Scheits (1703). Image obtained from the Picryl website.[16]

Modern formal logic is often identified with mathematical logic. However, such a judgment cannot be accepted in its entirety. Only a part of formal logic related to studying mathematical artifacts can be strictly recognized as mathematical. At the same time, modern logic contains many sections and branches that operate with other concepts, relate more to philosophy than to mathematics (modal logic, inductive logic, multivalued logic, logic of norms and evaluations, etc.), and, consequently, apply the verbal form of hypotheses and conclusions.

The interpretation of probability as a degree of reasonable belief, which was proposed by the English scientist John Maynard Keynes in the 1920s, is considered the first interpretation that largely contributed to the foundation of structures where the concepts of probability and logic converge. Analyzing and criticizing classical and frequentist interpretations of probability, J. Keynes proposed to consider probability as the degree of a reasonable belief attributed to relevant statements (events) given exactly fixed data. "The terms *credibility* and *probability*," wrote J.M. Keynes, "describe the different degrees of reasonable belief in some statement which we can attribute to that statement under different knowledge."[17(p.1)] In Keynes's interpretation, probability (as some numerical value) is established for a logical relation between two sets of statements. It reflects the degree of belief in the correctness (or incorrectness) of a statement, and the numerical value of this degree of belief is not constant but varies depending on the available knowledge (available information) regarding the analyzed phenomena, objects, and processes. All of the above points to the analytical rather than empirical nature of probability. Notably, Keynes emphasized the objective nature of his interpretation of probability. Although the degree of reasonable belief (the value of probability) changes with changes in available knowledge, it characterizes a relationship between objects independent of human consciousness. "As an important category for logic," Keynes argued, "probability is not subjective. It is not subject to human caprice. A statement is probable not because we think so of it. Once the facts that determine our knowledge are provided, then, under those circumstances, what to consider probable and what to consider improbable is fixed objectively and does not depend on the subject's opinion. Probability theory is logical because it deals with the degree of belief that is reasonable under given conditions, not merely with factual belief, which may or may not be reasonable."[17(p.1)]

Attempts to build a rigorous system (theory) of probabilistic logic based on these conclusions did not succeed.

What was done in this direction was too cumbersome and inconvenient. The described ideas have not found wide direct application in practical activity either. At the same time, we note that the work in this direction was not in vain. They contributed to realizing the connection between logic and probability, which led to the formation of new, practically useful scientific directions. Today, the scientific community perceives the degree of reasonable belief not as the actual belief of the subject in the corresponding statement, but as a logical relation between statements. In this case, the measure of reasonable belief is transformed into the degree of confirmation, which is the foundation of the modern theory of inductive logic.

7. Multivalued logic

Classical logic, by definition, cannot operate with concepts that are delineated concretely, ambiguously, because all statements (logical variables and hypotheses) in the formal systems of this logic can have only two mutually exclusive states: "true" with truth value "1" or "false" with truth value "0."

The principle of ambiguity (the law of the excluded third) determines the uncompromising discreteness of the representation of analyzed phenomena, leaving no room for any uncertainty, modal, and probabilistic assessments of these phenomena. However, in many cases, the essence of any statement, conclusion, or hypothesis can be perceived with more or less skepticism. After all, the statement "not full" in many cases does not mean "empty," just as the statement "not empty" may not fully characterize an object as "full." Binary assessment of truth in such situations imposes significant limitations on the realization of relevant studies and conclusions. It begins to hamper and restrain the activity of both the creators of scientific theories and the engineering community. The need to reflect the uncertainty, and incompleteness of the concepts of "truth" and "false" in the process of solving professional tasks of specialists from various fields of knowledge led to the search for, and eventually to the emergence of appropriate mathematical artifacts that significantly expanded the boundaries of bivalent logic, turning it into multivalued logic as a kind of formal logic that allows the existence of more than two values of a logical variable.[18,19]

One of the first attempts to move beyond binary logic was made in 1920 by the Polish philosopher Jan Lukasiewicz, who introduced a three-valued logic. Within this framework, truth values were expanded beyond the traditional "true" and "false" to include an intermediate value: "possible." These values corresponded to the grades "1 (true)," "0.5 (possible)," and "0 (false)."

Infinite-valued logic for two variables whose probabilities of truth are *a* and *b is* introduced by formal statements: the true value of a variable is characterized by the value being on the interval of real numbers from 0 to 1, negation is defined as *1-a,* conjunction as *min* (*a*, *b*)*,* and disjunction as *max* (*a*, *b*)*.*

The introduced rules made it possible to describe processes and phenomena as functional dependencies between the factors acting on the object (function arguments) and reactions of this object (function values), carry out identical transformations of these dependencies, and form appropriate solutions. Multivalued (infinite-valued) logic with a certain degree of conventionality can be considered a forerunner of fuzzy logic.

8. Probability and infinite-valued logic

At first sight, the theory of probabilities appears to be analogous (or at least very close in content) to infinite-valued logic, in the framework of which the truth of realization of **some** event is characterized by the category of possibility, that is, it is established with some probability, which takes the value from 0 to 1 (zero truth means complete falsehood, one truth means complete certainty). The probability of non-realization of some event corresponds to negation, while the probability of simultaneous realization of two events corresponds to conjunction.

However, there is a fundamental difference between probability theory and infinite-valued (and multi-valued) logic. In logic, the true value of any function is completely determined by the true value of its arguments, whereas in probability theory, the probability of a complex event (integrated from component events) depends not only on the probabilities of the component events but also on their mutual dependence, which is determined through the corresponding conditional probabilities. This is manifested, in particular, in the fact that in probability theory, the equivalent of the "law of the excluded third" is fulfilled. In probability theory, "some event is realized or not realized" is always equal to one (there is no third option), whereas, in infinite-valued logic, the law of the excluded third is not fulfilled. Similarly, in probability theory, the equivalent of the "law of contradiction" is fulfilled: the probability that "some event is simultaneously realized and not realized" is always equal to zero, whereas, in infinite-valued logic, the law of contradiction is not fulfilled.

At the same time, an overly critical analysis allows us to establish the existence of some connection between the truth values of the categories of infinite-valued logic and the probabilities of these categories (given the classical interpretation of the notion of probability). This connection is manifested in the above provisions, which formally introduce infinite-valued logic and can be formulated as probabilistic conclusions. Specifically, if *a is* the probability of realization of some event, then the probability of not realizing the same event will be defined as (*1-a);* if *a* and *b are probabilities of realizing some two events,* then the probability of joint (simultaneous) realization of these two events will not exceed the value *min* (*a*, *b),* the probability of realizing at least one of the two events will not exceed the value *min* (*a*, *b),* and the probability of realizing *at* least one of the two events will not exceed the value *min* (*a*, *b).*

The connection between the concepts of probability and logic is clearly shown if we interpret the probability of the realization of an event as a *measure of confidence in* such realization. In this interpretation, the probability of realization of an event is considered as a measure of the subject's belief in this realization, that is, the fact of realization of the event is "put" into the logic of thinking of the subject, who, of course, in his conclusions relies on the relevant available knowledge, or his intuition and experience. An event, in the realization of which there is no full belief but with some assumptions about the possibility of such realization (the possibility of realization is not rejected completely), is considered probable; the value of the specified probability characterizes its truth. Given that the probability value in this interpretation relies on the degree of belief of the subject, this interpretation of probability is called *subjective probability.*

In an explicitly developed form, the English scientist Frank Ramsey first presented such an interpretation of probability in his book *Fundamentals of Mathematics*, published in 1920. The interpretation of probability as a measure of confidence is convenient because it directly indicates the practical method of calculating probability. To realize this method, Ramsey offered the analysis of the betting process by some subjects regarding the realization of event *A*. If, in the process of betting, the subject estimates the probability of the realization of event *A* based on appropriate knowledge, logic, experience, and intuition, to the highest bet, then this bet is offered by this subject. For example, if the bet 3:2 was accepted, the subject estimates the probability of the realization of event *A* as 3/5. Although this method has disadvantages, it has become a foundation for further development in building probabilistic conclusions.

The interpretation of probability as a measure of confidence, that is, introducing the concept of subjective probability, raises at least two questions. *Firstly, can the concept of subjective probability be applied in a practical activity?* Today, this question has a quite concrete answer: wherever there is a need for a pragmatic assessment of the realization of some event, we inevitably encounter the categories of subjective probability. The concept of subjective probability has significant applications in modern decision-making theory. *Secondly, what is the place of subjective probability among other interpretations of probability?* Some scientists consider subjective probability the only possible one, the basis of all other interpretations. However, if one accepts such an interpretation of probability as the only true and admissible one, one cannot rationally explain the existence of the stability of frequencies of mass phenomena, which is the basis for the introduction of an objective (empirical) interpretation of probability. Besides, based on the concept of subjective probability, one cannot explain the successful practical applications of the concept of statistical probability in various fields of human activities, including forecasting procedures.

9. Probabilistic logic

Probabilistic logic is a section of logic that studies logical structures for which the set of truth components of statements consists not of two values: "true" and "false," as in binary logic, but of all real numbers placed on the interval [0, 1]. In this case, as in binary logic, zero corresponds to an impossible event, and one to an event that occurs reliably. Truth values in probabilistic logic are called truth probabilities, degrees of likelihood, or degrees of confirmation.[18,19] Probabilities are added to the system of binary logic as a new relation connecting the set of statements (variables) and the set of truth values of these statements (the set of values of the corresponding functions) from the interval $0 \leq P(A) \leq 1$, where $P(A)$ is the probability of the truth of the statement (event, variable) *A*. Probabilistic logic is often considered a refinement of inductive logic because the relationship between the premises of an inductive judgment can be evaluated using a probability, the value of which can be determined numerically or by comparing the probability of the truth of a statement (event, variable). Today, the importance of probabilistic logic in scientific research and practical activities is continuously growing due to the significant development of information systems, artificial neural networks, and robotics.

One of the varieties of probabilistic logic systems is *systems of pragmatic probabilistic logic,* in which probability is applied to analyze pragmatic aspects of research. Such logical systems include the probabilistic logic of action, probabilistic logic of choice, probabilistic logic of change, *probabilistic logic of decision-making, probabilistic logic of preference, probabilistic logic of attractiveness, and probabilistic logic of priority. In* some of these systems, the concept of probability does not appear explicitly, but its connection with the basic concepts of the corresponding structure in each case is traceable.[18,20] If the realization of a random event from some statistical set of events is assigned a certain probability, then the same probability can be correlated with a judgment characterizing this event, that is, an unambiguous correspondence between events and judgments concerning these events can be established. Probabilistic logic, however, relies on a logical interpretation of probability, according to which this probability is considered as a relation between the premises and conclusions of induction. The first systems of probabilistic logic arose precisely in the framework of the logical interpretation of probability, so logical probability is often defined as an *inductive probability.*

The substantive essence of probabilistic logic consists in the fundamental impossibility of constructing certain logical structures to avoid, and completely exclude from consideration, the category of *incomplete reliability (relative truthfulness*) of premises and conclusions, which is inherent in any knowledge based on inductive conclusions, that is, any inductive knowledge. As a logical system, probabilistic logic is a type of infinite-valued logic. The multivaluedness (or rather infinite-valuedness) of the probabilistic evaluation of a hypothesis does not deny the fact that the hypothesis itself can have only one of two truth values ("true" or "false"). Probability remains binary, because the value of probability characterizes the relation of a given hypothesis to reality not directly (the direct relation of a hypothesis to reality remains two-valued), but indirectly, through other hypotheses based on the available information (knowledge) concerning the events being analyzed, that is, on the information that is available at the present moment. Hence, the *concept of probabilistic logic treats probability as a characteristic of a logical variable.*

Rudolf Carnap first outlined the theory of probabilistic logic in its most complete form in his work *Logical Foundations of Probability*. In Carnap's interpretation, the concept of probability is *considered a certain category of inductive logic. In this case, probability characterizes the logical connection between judgments and the* degree of confirmation of *hypothesis H* by the *data E*; in *other words, relative to the data E, hypothesis H* has *a probability* P. *In his work, Carnap argues that* such probabilistic-logical structures should be considered analytical because they say nothing about the environment, and are independent of the empirical truth of both *E* and *H*. However, these categories can be determined predominantly by empirical data.

Note that within the framework of this concept, the concept of probability has nothing in common with the concept of truth. Attributing to hypothesis *H, the* degree of probability $P = 1$ with respect to certain data *E does not mean the truth of this* hypothesis, because the data *E* itself may be false, and under such conditions, the truth of hypothesis *H cannot be* deduced based on logical conclusions. For the same reason, the probability $P = 0$ does not mean that hypothesis *H* is false with respect to the data *E.*

The validity of the numerical determination of the probability of some statements based on other statements is currently a subject of discussion. It is solved differently by representatives of different branches of probabilistic logic. However, the calculation of the probability of complex hypotheses (complex logical functions), which provided that the probabilities of the individual components of such hypotheses are known, in all systems of probabilistic logic is carried out according to the rules of mathematical calculus of probabilities, which today is based on the axiomatic system. Such a system defines the properties of abstract categories for which probabilistic reasoning is applicable, as well as the corresponding rules for obtaining the probabilities of certain categories (taking into account the formation of their corresponding structure) based on the probabilities of others. The probabilistic-logical concept fits into the field of logic and acquires its essence when analyzing (based on logical conclusions) the connections between individual hypotheses.

10. Logical probability (probability logic)

Logical probability is a logical relation between two hypotheses (events, variables, and statements), the characteristic feature of which is that this relation establishes the degree (quantitative value) of confirmation of some hypothesis *H by* evidence *E*. The subject of probability logic is the calculation of the probability of the truth of random events that take exclusively two values: 1 or 0.[21,22]

To provide a mathematical interpretation of the above definition of logical probability, consider some logical function f (x), which can take a true value on a set of data x. It is known that the acceptance of a true value by the function f (x) can occur only with a certain probability *R,* which is the corresponding logical probability. Mathematically, it is written as in Equation II.

$$P\{f(x) = 1\} = R \tag{II}$$

The above should be interpreted as follows. There is relevant information regarding the truth of a set of variables x. Based on this information, we should establish the probability of the logical function f (x) taking a true value. Of course, the more information there is about the truth of variable x, the higher the value of the specified probability.

Regarding the statistical interpretation of probabilities, conditional probability is the function for confirming the truth of a proposed hypothesis. Probabilistic logic and logical probability are not synonyms, but essentially different categories and concepts for assessing the truth of the corresponding hypotheses. In the first case, we deal with infinite-valued (multi-valued) logic; in the second, we deal with two-valued, binary logic.

To emphasize the difference between "probabilistic logic" and "logical probability," the notion of *deterministic* logic, essentially a binary logic, is introduced. With the help of deterministic logic, it is proposed to construct complex logical functions based on appropriate logical variables or the convolution ("aggregation") of the simplest logical functions. Substitution of logical arguments by probabilities of their truth is further realized in the obtained logical function by special algorithms. The possibility of applying these procedures was emphasized by Andrey Nikolaevich Kolmogorov, the founder of axiomatic probability theory, who[13] proved a theorem called the *law of "0 and 1" (the law of "zero"/"one").* According to this theorem, in the case when the marginal probabilities are known and are necessarily equal to zero or one, all formulas of probability theory for complex logical functions f (x), whose arguments are *events independent in the aggregate,* become correct logical formulas when replacing the specified events with the corresponding statements. We should consider that the law of "0 and 1" assumes mutual independence of all events of the analyzed totality. We can say that the variables of the analyzed population must be orthogonal. The orthogonalization of the space of variables is the most computationally complex procedure that accompanies the practical application of logical probability.

Specific values of logical probability for each synthetic argument depend on another judgment, which can be interpreted as a description of a subject's knowledge (a description of the information available to this subject) of the synthetic argument being analyzed. Given this interpretation, logical probability is often referred to as epistemological probability, that is, the probability that depends on the available knowledge. Under certain conditions, logical probability can be treated as *subjective probability.* However, the value of logical probability is unambiguously determined by a given system of knowledge and available information, and hence, has an objective character. Since logical functions describe some event (for example, the state of a device in the corresponding technical state), logical probability can be considered a function that determines the corresponding characteristic of such events.

It should be noted that the concept of logical probability is the basis of the logical probabilistic method of analyzing structurally complex systems, which is currently used to establish indicators of failure-free, safety, and survivability of such systems.

11. Fuzzy sets and fuzzy logic

Probability theory aims to solve problems in which the event under study is described by inaccurate information. In this case, for the correct application of probability theory, the frequency of realization of the event under study $v = n/N$ (n is the number of realizations of the event according to the results of N observations) must tend to stability with increasing value A., that is, statistical stability should be observed. If the specified stability is observed, then n is a random variable that will characterize the measure of the inaccuracy of information regarding the event under study. However, if the specified stability is not observed, then P cannot be considered as a measure of the inaccuracy of the specified information. This means that probability theory and methods of mathematical statistics cannot be applied to research. At the same time, it is known[23,24] that the *source of the inaccuracy of information is not only its probabilistic nature but also its fuzziness.* The theory of fuzzy sets (fuzzy theory) was developed by Lotfi Zadeh to precisely account for the inaccuracy associated with fuzziness.[23]

A mathematical theory of fuzzy sets allows us to describe fuzzy concepts and knowledge, operate with this knowledge, and make corresponding fuzzy conclusions, often providing sufficient information for making adequate and effective decisions in various areas of human activity. Fuzzy sets can also be formed by corresponding logical variables, for which *in the* space of fuzzy sets, it is possible to establish quite specific rules, in many respects similar to the rules of conventional binary logic.

The main idea of introducing fuzzy sets is based on realizing that real human judgments, based on appropriate linguistic constructions, cannot be described within the framework of traditional mathematical artifacts. As an alternative, fuzzy sets, classes of variables with imprecisely defined boundaries, are proposed. Such classes are described by belonging functions, which characterize the degree of belonging of the corresponding variable to a certain class (the degree of certainty that the variable belongs to this class). At the same time, these belonging functions can be regarded as fuzzy models of corresponding events, conclusions, and hypotheses. The construction of such models usually relies on expert evaluations, heuristics, and/or approximation methods of generalization of experimental data. The above and the development of appropriate mathematical operations on such structures provided the basis for developing a new direction in modeling various objects, phenomena, and processes, especially when numerical and linguistic variables describe their properties.

The place of fuzzy theory among other mathematical theories, as well as the objective need for practice in this theory, is best reflected by its author Zadeh:

The term fuzzy logic is currently used in two different senses. In a narrow sense, fuzzy logic is a logical system that aims at a formalization of approximate reasoning… In a broad sense, fuzzy logic is almost synonymous with fuzzy set theory. Fuzzy set theory, as its name suggests, is basically a theory of classes with unsharp boundaries. Fuzzy set theory is much broader than fuzzy logic in its narrow sense and contains the latter as one of its branches.[25(p.78)]

As noted above, a characteristic feature of axiomatic probability theory is that the objects analyzed in the framework of this theory form a set *S* of unit measure *P* (*S*) = 1. Such a measure in mathematics, as is known, is called a *probability measure.* However, other sets of objects and measures correspond to these sets that have remained unattended by probability theory. In particular, there are known sets of objects for which the measure can be greater or less than one. Such sets constitute the subject of fuzzy set theory. Usually, these measures are considered as a measure of *possibilities* (P (S) > 1) and a *measure of necessities* (N (S) < 1). It is quite clear that by simultaneously considering all three measures (P, *P*, and *N*), we go beyond the limitations of probability theory, cover the whole space of objects, information about which, under certain conditions, can be provided in incomplete, inaccurate, and non-numerical form.

As it has been repeatedly mentioned above, the existence of statistical stability of the results of observations related to establishing the frequency of realization of the corresponding event among a set of events of a certain class can rarely be guaranteed and practically cannot be verified. Nevertheless, despite these methodological difficulties, today, specialists mainly use well-developed and widely known methods of probability theory to interpret the results of such observations. In many respects, this situation is promoted by the current system of training specialists of corresponding (mainly engineering) specialties, in which too little attention is paid (and often not at all) to the theory of fuzzy sets, but a detailed study of probabilistic methods and practical calculations based on these methods is assumed, often without a thorough analysis of the limits of their application.

There is a significant difference between probability theory and fuzzy set theory concerning the requirements related to the accuracy of the raw data and the conclusions drawn from the data. By definition, probability theory is adapted to handle large data sets from which conclusions are drawn. The theory of fuzzy sets is adapted to "work" with a significantly smaller amount of data, allowing their incompleteness, inaccuracy, and non-numerical nature (and simultaneously, within the study of one object). This distinction leads to greater stability of the results obtained based on fuzzy set theory when there are little data to assess what is happening. In many cases, this becomes the main factor in choosing fuzzy algorithms as a tool for solving practical problems.

Based on the introduced concept of fuzzy sets, it was reasonable to generalize classical binary logic based on consideration of an infinite set of truth values, that is, to pass to fuzzy *logic.* In the proposed version of fuzzy logic, the set of truth values of statements is generalized to the interval [0, 1], that is, it includes binary logic and finite multi-valued logic (including three-valued Lukasiewicz logic) as special cases. Thus, it becomes possible to consider proposed hypotheses (logical variables) with different truth values, to rely on a small amount of available information, to give experts some freedom in assessing the situation, and to form conclusions with some, but for most applications with quite acceptable uncertainty, which, moreover, quickly decreases in the process of receipt of additional information regarding the proposed hypotheses.

12. Probability, logic, and fuzziness as engineering tools

According to the classical interpretation, probability is the ratio of the number of selected realizations of some experiment to the number of all its possible realizations. At the same time, all of these realizations are considered equally possible. Criticism of the shortcomings of this definition led to the emergence of the concept of frequency probability (based on which mathematical statistics emerged), and later to the axiomatic theory of probability. Today, probability theory and mathematical statistics are well-developed and comprehensively understood. Despite certain difficulties in substantiating the corresponding algorithms and interpreting the obtained results, both of these tools (both probability theory and mathematical statistics) are widely used in the processes of realizing the life cycle of technical objects, in social research, in the construction of artificial intelligence systems, and in general, all situations where the available information is sufficient to obtain acceptable adequacy of conclusions. The measure of adequacy becomes the values of confidence intervals and calculations. Among the tasks that are successfully solved based on the methods of probability theory and mathematical statistics, the following are particularly noteworthy: processing the results of research, measurements, and control of parameters of various objects by nature (both in engineering practice and in other areas of human activity), calculations of reliability indicators of technical objects (although, mainly predicted reliability at the design stage), analyzing the reliability of technical objects (although, mainly predicted reliability at the design stage), analyzing the reliability of technical objects, and analyzing the results of the analysis of the reliability of technical objects.

At the same time, it should be clearly understood that probability theory and mathematical statistics algorithms will only work if the primary information on the analyzed event is sufficient and if additional conditions for implementing relevant studies are met. After all, these algorithms aim to process *mass phenomena and arrays of information. Additionally, they have limitations in the form of appropriate conditions that ensure the stability of the results of calculations, but they* can neither be strictly proved nor tested in practice. If the available information is limited, the events under study are not sufficiently numerous, the uniformity of the conditions for the realization of these events in the process of observation is not ensured, or there are methodological difficulties with the calculations of probability indicators and the corresponding confidence intervals – especially, when the value of these intervals is too large – then the results obtained may fail to reflect reality accurately. Under such conditions, searching for other methods and algorithms to solve the set tasks is necessary.

The interpenetration of the categories of logic and probability opens up a space for logical interpretations of probabilistic concepts and actualizes, and justifies the possibility of determining the prior value of probability through the study of the space of possibilities. At the same time, such possibilities can be defined as unequal, with different weights.

Probabilistic logic is logic whose subject matter is probabilistic variables (statements, hypotheses, and judgments), regardless of whether probability is considered as a property of an individual variable (under such conditions, probability is attributed to a given variable as an intermediate value between truth and falsity, i.e., within the range from zero to one) or as an evaluation of the relation of "ordinary" binary variables. Unlike in probability theory, labeling probability by an exact number (defining probability by a particular quantity) is not a major requirement in probabilistic logic. The logical apparatus built on such a basis is used for the approximate evaluation of hypotheses not by correlating them with reality, but through other categories that reflect the available knowledge (available information) about the objects, phenomena, and processes under study.

Logical probability relies on the available body of knowledge (available information) and does not depend on unknown facts. In contrast, frequency probability relies on facts existing in the real world of relationships of these or those events and has no relation to the available body of knowledge.

Today, methods that rely on structural-logical schemes of such objects are widespread in the practice of calculations of fail-safety, analysis of emergencies, and construction of algorithms for diagnosing structurally complex technical objects. In this case, all signals available in the object (in the broad meaning of the word, i.e., outpourings and reactions of different origins, such as electrical, mechanical, and hydraulic) are supplied by binary logical variables. The technical state of the object is identified by the corresponding logical function that actualizes the study and development of the concept of logical probability and methods of logical-probabilistic calculus, based on this concept.[26,27]

In realizing the life cycle of any technical object, we often have to face a situation when the available data are insufficient for applying probability theory methods and algorithms associated with logical interpretations of probabilities. One key task is establishing and aligning the nomenclature and quality standards between the executor and the customer for the object proposed for development. This involves considering and comparing various potential scenarios for the future realization of the specified object based on the generalized quality indicator. In addition, it includes forecasting the "residual" levels of reliability and safety indicators during the object's operation, an especially critical task for long-term-use objects. The main advantage of such methods is the possibility of obtaining an adequate answer to the posed questions concerning the analyzed object, process, or event in the presence of limited, incomplete, and inaccurate prior information, which presented in numerical and non-numerical forms. Notably, the amount of information sufficient for the initial idea about the properties of the analyzed structures can be obtained based on expert evaluations (providing the experts with sufficient comfort regarding the formulation of conclusions), comparison of prototypes, experience, and intuition of specialists engaged in solving the set tasks.

13. Conclusion

Probability theory is deeply and comprehensively developed. Despite certain methodological difficulties regarding the compliance of the real data set in the research space with the basic provisions, the probability theory allows for obtaining adequate results based on which effective decisions can be made, which provided that there is a sufficient amount of these data. It should be recognized that the degree of confidence in the results obtained can be assessed by calculating the appropriate confidence intervals.

However, the application of probability theory is often associated with the need to make poorly justified (or not justified at all) simplifications of the analyzed object (process) model, which significantly reduces the value of the obtained results and casts doubt on their adequacy. In addition, applying probability theory becomes practically impossible when the primary data volume is insufficient. It should be noted that in many cases (for example, in the process of calculating the failure rates of an object), the direct application of probability theory algorithms does not take into account the structure of the object under study, and is also burdened by the condition of conformity of the mode of functioning of the components to the modes of establishing the corresponding basic indicators (e.g., failure rates of components). Fundamentally, differences in the structure of objects operating under the influence of various sets of factors with differing intensities should be represented by the same models for applying probability theory algorithms. This approach can result in significant methodological errors in the obtained results.

Logical interpretation of probability (in particular, probability logic) overcomes some of the methodological difficulties of probability theory. Logical probability allows us to consider the structure of the analyzed object, identify potentially dangerous places in such a structure, and consider the possibility of the occurrence and spreading of an emergency. It also provides additional opportunities to improve the efficiency of diagnostic algorithms. However, relying on statistical material in calculating the final results of the study of probability logic is tightly bound to the availability of appropriate statistical data. Obtaining such data is often realized in conditions that do not correspond to the real conditions of the object's operation. This can significantly distort the obtained conclusions, reducing the effectiveness of appropriate solutions.

Fuzzy set theory is adapted for processing small amounts of incomplete and inaccurate information, which can be represented in numerical and non-numerical forms. The corresponding methods and algorithms are usually suitable for processing expert opinions, as well as recommendations and suggestions based on the experience and intuition of specialists. At the same time, these experts and specialists can freely and comfortably formulate their assessments numerically and linguistically. The theory of fuzzy sets is most suitable for establishing "current" indicators of operational reliability and residual resource, as well as the safety and survivability of structurally complex objects of long-term use.

At the same time, it should be noted that the methods of fuzzy set theory are not able to fully replace the methods and algorithms associated with specific calculations, which are based on sufficient volume and carefully verified statistical material, and are successfully implemented in the framework of probability theory and its logical interpretation.

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