

The smoothed finite element method (S-FEM): A framework for the design of numerical models for desired solutions

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ABSTRACT The smoothed finite element method (S-FEM) was originated by G R Liu by combining some meshfree techniques with the well-established standard finite element method (FEM). It has a family of models carefully designed with innovative types of smoothing domains. These models are found having a number of important and theoretically profound properties. This article first provides a concise and easy-to-follow presentation of key formulations used in the S-FEM. A number of important properties and unique features of S-FEM models are discussed in detail, including 1) theoretically proven softening effects; 2) upper-bound solutions; 3) accurate solutions and higher convergence rates; 4) insensitivity to mesh distortion; 5) Jacobian-free; 6) volumetric-locking-free; and most importantly 7) working well with triangular and tetrahedral meshes that can be automatically generated. The S-FEM is thus ideal for automation in computations and adaptive analyses, and hence has profound impact on AI-assisted modeling and simulation. Most importantly, one can now purposely design an S-FEM model to obtain solutions with special properties as wish, meaning that S-FEM offers a framework for design numerical models with desired properties. This novel concept of numerical model on-demand may drastically change the landscape of modeling and simulation. Future directions of research are also provided.

KEYWORDS computational method, finite element method, smoothed finite element method, strain smoothing technique, smoothing domain, weakened weak form, solid mechanics, softening effect, upper bound solution

1 Introduction

1.1 Brief history on the finite element method (FEM)

As one of the most successful numerical methods, the FEM [1–4] has been well developed and now widely applied to solve mechanics problems in sciences and engineering, including structural analysis and design, material design and evaluation, fluid flows, thermodynamics, soil mechanics, biomechanics, electromagnetism, just to name a few. The key ideas and techniques of the FEM were mainly established, fine-tuned and perfected over 1950s–1980s. The democratization and popularization of FEM are largely supported by three important factors: solid mathematic theory, fast development of computer hard-

ware, and development of user-friendly commercial software packages. Despite the huge success, the standard FEM, however, has also some limitations, including Refs. [5,6]:

1) Poor accuracy in stresses when a linear triangular (2D) or tetrahedral (3D) mesh, or T-mesh, is used. This is due to the overly-stiff behavior rooted at the fully-compatible element-confined Galerkin weak formulation using assumed displacement field. The T-mesh is, however, the simplest and the only mesh type that can be generated automatically for complicated geometries of solids and structures. Hence, it is indispensable mesh to use in practice. Efforts to make T-mesh usable effectively are thus extremely important, and much of the efforts on meshfree methods are with this goal [7,202–204].

2) The standard FEM demands for high quality mesh, when quadrilateral (Q4) and hexahedral (H8) elements are

used. This often leads to time-consuming and costly manual operations by the analysts. In addition, sophisticated software packages are needed in the creation and checking the quality of the meshes.

3) Mapping is a must in the FEM to ensure the compatibility on the interfaces of Q4 and H8 elements. This not only leads to higher computational costs for the evaluations of the Jacobin matrix, but also poor solutions and even breakdown during computation when an element is heavily distorted. This is because the Jacobin matrix can become badly conditioned. For this reason, the analysts need to be properly trained in using a sophisticated pre-processor for creating FEM models.

4) The fully compatible FEM solution is always a lower bound (for force-driving problems, in strain energy measure). Lacking of upper bound leads to difficulty to quantify the solution errors, and to determine the necessary mesh density. Trial-and-error is often required.

5) Volumetric locking phenomenon: The solution error increases significantly for incompressible solids whose Poisson's ratio closes to 0.5, at which the bulk modulus approaches to infinite and thus dominates the strain energy of the entire system, leading to erroneous solution that is "locked" in volumetric behavior.

Alternative theory and techniques are needed to address the above-mentioned problems, a reliable numerical method that works effectively with the most simplicial T-mesh can be critically valuable.

1.2 Brief history on the smoothed finite element method (S-FEM)

The S-FEM [8–36] was originated by G R Liu and coworkers based on finite element mesh by applying the strain smoothing techniques [37] that was used to stabilize the nodal integrated Galerkin meshfree methods, and the concept of point interpolation method (PIM) used to construct meshfree shape functions [5]. The first paper of S-FEM was published in 2005 [9] using smoothing cells created based on nodes, but it was termed as LC-PIM as a part of efforts on meshfree method development using PIM. When linear PIM is used, the LC-PIM is actually the NS-FEM using linear 3-noded triangular (Tr3) elements. In 2008, the 3D version of NS-FEM (was still termed as LC-PIM) using 4-noded tetrahedral (Te4) elements was developed [10]. In 2008, the important upper-bound-solution and volumetric-lock-free properties of NS-FEM was discovered and examined in detail [11]. Since then a family of models have been established through a number of innovative constructions of smoothing domains (SDs). The S-FEM becomes a valuable combination of the standard FEM [4] with the strain smoothing operation [37] using innovative types of SDs [8] and PIM used in the meshfree techniques [5]. This novel combination effectively addresses almost all the limitations in the FEM. Typical S-FEM models developed so far are:

1) Cell-based S-FEM (CS-FEM) for both 2D and 3D problems [12,15,46,165,169,171,174,177,182,193]. The first CS-FEM paper was published in 2007 [12];

2) Node-based S-FEM (NS-FEM) for both 2D and 3D problems [20,41,99,166,176,194]. The first paper on NS-FEM was published in 2005 [9];

3) Edge-based S-FEM (ES-FEM) for 2D and 3D [21–24,33,47,61,62,72,75,77,82,88,110,129,141,172,192,197,210]. The first paper on ES-FEM was published in 2009 [21];

4) Face-based S-FEM (FS-FEM) for 3D [27–29,94,164,195]. The first paper on FS-FEM was published in 2009 [27];

5) Hybrid types of smoothing domains was also designed, in constitutive-matrix based selective manner [16,29,32–35,48–56,74,76,157,187], and in domain-based manner: S-FEM/FM-BEM [188], sub-domain S-FEM [189], ES-XFEM [85,89], α FEM [18,25,26,48,49,168,178], β FEM [36,198], etc..

Studies have found that each S-FEM model can have different properties or unique features, depending on the types of smoothing domains used [8]. These S-FEM models have already been applied to a wide class of mechanics problems of solids and structures, including:

1) Stress analysis and design for structures [8,13,163,186];

2) Vibration and dynamic analysis of various types of structures [14,47,48,67,71,72,100–108,166,176–178,206];

3) Hyperelasticity and biomechanics [35,56,156–162,187];

4) Elastic-plastic analyses [23,57,205];

5) Visco-elastoplastic analysis [24,28];

6) Contact analyses [50,51,58,59,165];

7) Heat transfer and thermo-mechanical problems [128–140,169];

8) Plates and shells [19,54,61–71,172,173];

9) Composites [68,70,72–74];

10) Structural or vibro-acoustics [109–124];

11) Limit and shakedown analyses [75,76,185];

12) Fracture mechanics, crack propagation, and fatigue [31,32,34,77–97,174];

13) Crystal plasticity modeling [36,60];

14) Stochastic analyses [98,99];

15) Impact problems [125];

16) Piezoelectricity and photonic devices [141–146];

17) Fluid-structure interactions [30,147–155];

18) Adaptive analyses [51,76,90,126,127];

19) Fluid dynamics [171] [175];

20) Porous media [182];

21) Topology optimization [142,143,184].

A detailed review on these models and their applications of S-FEM are referred to a recent review article [167]. Theoretical studies on stability and convergence for S-FEM can be found in [180,181,183]. Works on code development of S-FEM are available at [5,8,38,163,

170,179,181,190]. Some basic codes of S-FEM are available for free download at GRLab's website: <http://www.ase.uc.edu/~liugr>.

The formulation of S-FEM can be viewed as a typical weakened weak (W^2) formulation that were used for various types of problems [38–45,200]. The W^2 formulation consists of two layers of “weakening” treatments: one for the system equation in which the displacements are approximated and used in a Galerkin weakform, and another for the strain approximation based the assumed displacements via strain smoothing and using the Gauss divergence theorem. The former make the model stiffer known stiffening effects [4], and the later makes the model softer known as the softening effects [44,45]. Because the strain approximation uses only the assumed displacements and the geometric information of the smoothing domain boundary, no addition degrees of freedom (DoFs) are introduced. These two complimentary stiffening and softening effects enable the S-FEM a number of unique features listed as follows:

1) S-FEM models are theoretically proven softer than the FEM counterparts using the same mesh [5,8]. They are found often producing more accurate solutions, higher convergence rates, and much less sensitive to mesh distortion. If the S-FEM model is a stiffer model (such as the ES-FEM and FS-FEM), the solution is always more accurate than the FEM counterpart.

2) The NS-FEM is a softer model, and offers an efficient and practical means to produce upper boundary solutions [9–11]. Together with the lower bound solution of FEM, one can now bound the solutions from both sides, which is important for solution error quantification.

3) Combining the soft effects of NS-FEM and stiff effects of FEM in the formulation stage, one can create models (such as the α FEM) that can produce “close-to-exact” solutions [18,25,109,168,178].

4) S-FEM models work exceptionally well with Triangular (2D) or Tetrahedral (3D) mesh, or T-mesh, that can be automatically generated. This is because of the use of W^2 formulation [38,40]. They are thus ideal for automation in computations and adaptive analyses, and hence has profound impact on AI-assisted modeling and simulation. In addition, S-FEM uses also polygonal elements of arbitrary shape.

5) The NS-FEM is found naturally “volumetric-locking-free” [11,195,196], which is important for simulation nonlinear problems of soft materials, such as bio-tissues.

6) The S-FEM can be viewed as a typical case of W^2 model, and hence only the shape function values are needed in the formulation, and no derivatives of the shape functions are required. Therefore, no mapping (and hence no Jacobian matrix is involved) is needed in S-FEM. This Jacobian-free feature leads to insensitivity to mesh distortion, which is important for large deformation nonlinear problems.

7) Most importantly, one can in fact purposely design an

S-FEM model to obtain solutions with special properties. The S-FEM is a framework that enable the design of models for solutions with desired properties and unique features. This novel concept of numerical model on-demand may drastically change the landscape of modeling and simulation.

2 S-FEM formulations

2.1 Strain smoothing

Consider a problem of solid mechanics defined in problem domain Ω and bounded by Γ . The problem is first divided into a set of elements to form a mesh often in the similar manner as in the standard FEM. Because the S-FEM uses smoothed strains, one needs to create smoothing domains on top of the element mesh. The problem domain is thus divided further into a set of non-overlapping and non-gap smoothing domains Ω_k^s ($k = 1, 2, \dots, N_s$) bounded by Γ_k^s , such that $\Omega = \bigcup_{k=1}^{N_s} \Omega_k^s$ and $\Omega_i^s \cap \Omega_j^s = \emptyset$ for $i \neq j$, where N_s is the number of the smoothing domains. The smoothed strain can then obtained for point \mathbf{x}_C in a smoothed domain using

$$\begin{aligned} \bar{\boldsymbol{\epsilon}}_k(\mathbf{x}_C) &= \int_{\Omega_k^s} \underbrace{\boldsymbol{\epsilon}^h(\mathbf{x})}_{\text{compatible strains}} \underbrace{W_k(\mathbf{x} - \mathbf{x}_C)}_{\text{weight function}} d\Omega \\ &= \int_{\Omega_k^s} \underbrace{\mathbf{L}_d \mathbf{u}^h(\mathbf{x})}_{\boldsymbol{\epsilon}^h(\mathbf{x})} W_k(\mathbf{x} - \mathbf{x}_C) d\Omega, \end{aligned} \quad (1)$$

where $\boldsymbol{\epsilon}^h(\mathbf{x})$ is the compatible strain obtained using the assumed displacements by differentiation, and \mathbf{L}_d is a matrix of differential operators. It is given as follows, respectively, for 1D, 2D, and 3D problems.

$$\mathbf{L}_d = \frac{\partial}{\partial x} \text{ for 1D, } \mathbf{L}_d = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \text{ for 2D,}$$

$$\mathbf{L}_d = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \text{ for 3D.} \quad (2)$$

In Eq. (1), $W_k(\mathbf{x} - \mathbf{x}_C)$ is a weight or smoothing function that satisfies the positivity and unity conditions:

$$W_k(\mathbf{x} - \mathbf{x}_C) \geq 0 \text{ and } \int_{\Omega_k^s} W_k(\mathbf{x} - \mathbf{x}_C) d\Omega = 1. \quad (3)$$

The following Heaviside-type function is the simplest, and most widely used as the smoothing function:

$$W_k(\mathbf{x} - \mathbf{x}_C) = \begin{cases} 1/V_k^s, & \mathbf{x} \in \Omega_k^s \\ 0, & \mathbf{x} \notin \Omega_k^s \end{cases}, \quad (4)$$

where $V_k^s = \int_{\Omega_k^s} d\Omega$ is the volume (for 3D) or the area (for 2D) or the length (for 1D) of the smoothing domain Ω_k^s . Substituting Eq. (4) into Eq. (1) and then using the divergence theorem, we obtain the smoothed strains:

$$\begin{aligned} \underbrace{\bar{\boldsymbol{\epsilon}}_k}_{\text{constant in } \Omega_k^s} &= \frac{1}{V_k^s} \int_{\Omega_k^s} \mathbf{L}_d \mathbf{u}^h(\mathbf{x}) d\Omega \\ &= \frac{1}{V_k^s} \int_{\Gamma_k^s} \mathbf{L}_n(\mathbf{x}) \mathbf{u}^h(\mathbf{x}) d\Gamma, \end{aligned} \quad (5)$$

where \mathbf{u}^h is the assumed displacement vector, which is obtained using simply the PIM [5,6] in the S-FEM. When 3-noded triangular (Tr3) or 4-noded tetrahedral elements (Te4) are used, \mathbf{u}^h can also be computed using exactly the same way as in the FEM using the shape functions created based on the physical coordinate system [4]. $\mathbf{L}_n(\mathbf{x})$ is a matrix containing the components of the unit outward normal on the smoothing domain boundary Γ_k^s :

$$\mathbf{L}_n(\mathbf{x}) = n \text{ for 1D, } \mathbf{L}_n(\mathbf{x}) = \begin{bmatrix} n_x^s & 0 \\ 0 & n_y^s \\ n_y^s & n_x^s \end{bmatrix} \text{ for 2D,}$$

$$\mathbf{L}_n(\mathbf{x}) = \begin{bmatrix} n_x^s & 0 & 0 \\ 0 & n_y^s & 0 \\ 0 & 0 & n_z^s \\ n_y^s & n_x^s & 0 \\ 0 & n_z^s & n_y^s \\ n_z^s & 0 & n_x^s \end{bmatrix} \text{ for 3D,} \quad (6)$$

where n_x^s , n_y^s , and n_z^s are the unit outward normal components on Γ_k^s , respectively, on the x -, y -, and z -axis. It is shown in Eq. (5) that the strain is computed via integration rather than differentiation. This is the one “weakening” operation at the stage of strain approximation. We note here that when the smoothing domain shrinks to zero, we have

$$\bar{\boldsymbol{\epsilon}}_k = \lim_{\Omega_k^s \rightarrow 0} \frac{1}{V_k^s} \int_{\Omega_k^s} \boldsymbol{\epsilon}^h(\mathbf{x}) d\Omega = \boldsymbol{\epsilon}^h(\mathbf{x}_C), \quad (7)$$

which means that the smoothed strain becomes the compatible strain, implying that the FEM is in fact a

special case of S-FEM at the limit when all the smoothing domains approach zero.

Apart from using the Heaviside-type function, one may use linear smoothing functions of equilateral triangle shape, as long as Eq. (3) is satisfied. In such a case, the smoothed strain is calculated via domain integration rather than a boundary integration.

2.2 Creation of different types of smoothed domains

S-FEM models use smoothed strains that are computed using Eq. (5) and the smoothing domains created on top of an element mesh. The art of the S-FEM is creative ways to form different types of smoothing domains, leading to different S-FEM models.

In a CS-FEM model, smoothing domains reside within the elements. In the CS-FEM using 4-noded quadrilateral (Q4) elements, for example, typical smoothing domains can be created by dividing the elements into smaller cells, as shown in Fig. 1, where one Q4 elements is divided into 1, 2, 3, 4, 8, or 16 smoothing cells (SCs). Each of the smoothing domains is bounded by four line boundary segments. In most applications, four SCs for each element are often used. Use of one SC can be more efficient and sometimes can produce upper bound solutions, but may have the so-called “hourglass” instability. It is also possible to used 4 SCs for some elements and 1 CS for other elements in the mesh, but this idea has not yet been implemented and studied in detail so far.

From Eq. (5), it is clear that the smoothing domain is used to evaluate the smoothed strains, the displacement values on the smoothing domain boundary is used in such evaluation, and the displacement values are interpolated using the nodal displacements. Because of this, the smoothed strains for the smoothing domain relate to all the nodes of the elements that have contribution to the smoothing domain. Such elements are called “supporting elements” and such nodes are called “supporting nodes” of the smoothing domain. For any smoothing cell in a Q4 element, the supporting element is 1, and the supporting nodes is 4.

Note that when Tr3-meshes are used, the CS-FEM will be exactly the same of as the FEM counterparts. This is because the compatible strains in Tr3 element is constant, and any cell-based smoothing operation can only produce the same constant smoothed strain. Similarly, when Te4-meshes are used, the 3D CS-FEM will be exactly the same as the 3D FEM counterparts.

When 3-noded triangular (Tr3) elements are used, one may create smoothing domains based on edges, leading to an ES-FEM-Tr3 models, as shown in Fig. 2 for 2D problems. In this case, each edge has a smoothing domain, and the total number of smoothing domains is exactly the number of edges in the mesh. A smoothing domain for an edge that is inside the problem domain, the edge-based

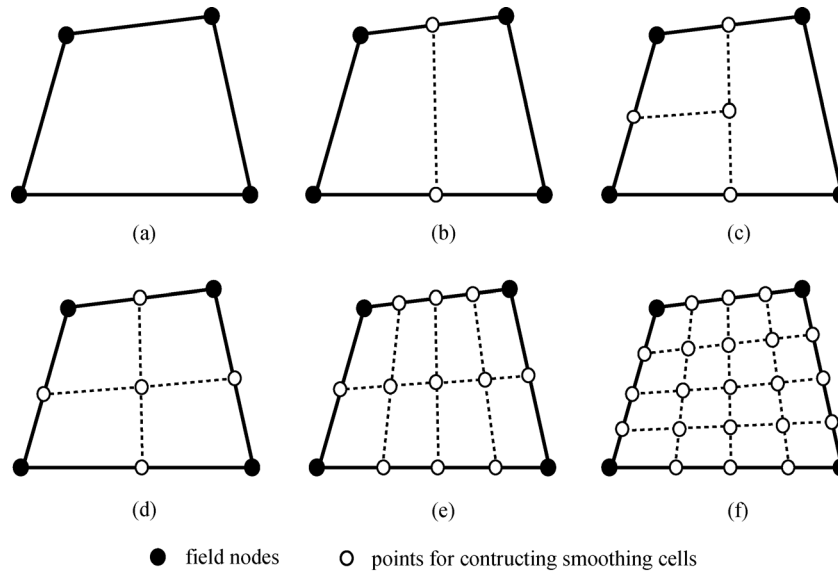


Fig. 1 Smoothing domains used in a CS-FEM model. A quadrilateral element may be divided into smoothing cells (SCs) by connecting the mid-segment-points of opposite segments of smoothing domains [36]. (a) $n_{SC} = 1$; (b) $n_{SC} = 2$; (c) $n_{SC} = 3$; (d) $n_{SC} = 4$; (e) $n_{SC} = 8$; (f) $n_{SC} = 16$ (from [98]). In applications, four SCs for each element are often used. Use of one SC can be more efficient and sometimes can produce upper bound solutions, but may have the so-called “hourglass” instability

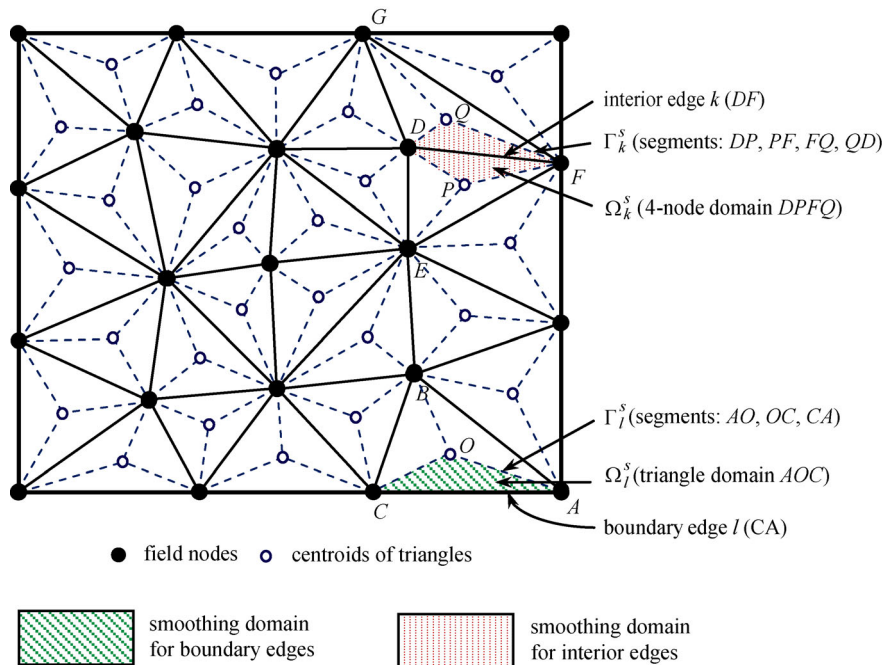


Fig. 2 Edge-based smoothing domains on a Tr3-mesh. Shaded areas are typical smoothing domains. The smoothing domain Ω_l^s is for edge l on the problem domain boundary, and is a triangle AOC for a boundary edge. Smoothing domain Ω_k^s is for interior edge k that is inside the problem domain, and it is a four-sided convex polygon $DPFQ$ (from Ref. [167])

smoothing domain is a four-sided polygon, the number of supporting elements is 2, and the smoothing domain boundary has 4 line segments. The number of supporting nodes of an interior edge-based smoothing domain is 4. For edge k , they are F , G , D , and E . For an edge on the problem domain boundary, and it is a triangle and the

smoothing domain boundary has only 3 line segments. The number of supporting element is 1, and the supporting nodes is 3 for a boundary edge-based smoothing domain. For edge l , these are A , O , and C .

An NS-FEM-Tr3 model uses node-based smoothing domains that are created based on nodes in a Tr3-mesh, as

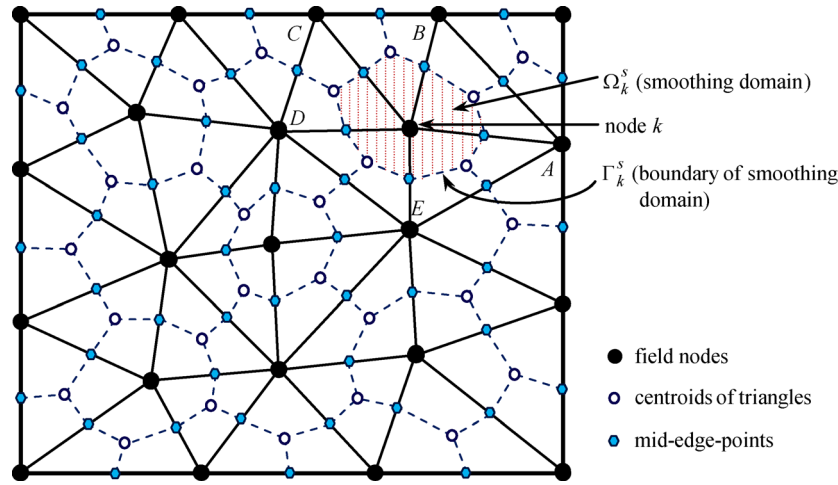


Fig. 3 Node-based smoothing domains for an NS-FEM model using Tr3-mesh. The smoothing domain Ω_k^s for node k is a polygon with $2n_k^e$ sides (where n_k^e is the number of elements surrounding node k) (from Ref. [167])

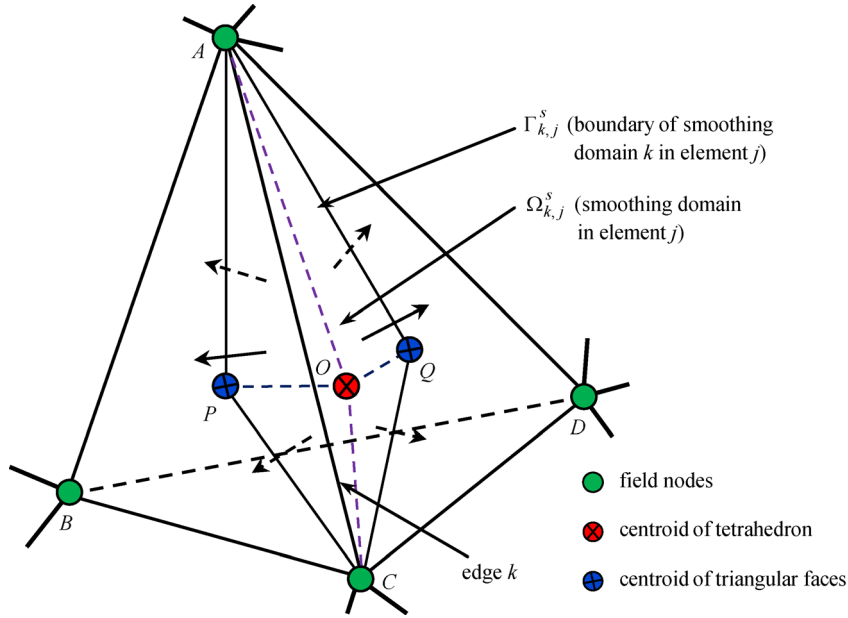


Fig. 4 Edge-based smoothing domain on a Te4-mesh for ES-FEM-Te4 model. Only the part of the smoothing domain $\Omega_{k,j}^s$ for edge k is shown. It is located inside element j , and is a double tetrahedron ACPOQ. If there are other elements connected to edge k , similar partial smoothing domains need to be constructed (from Ref. [167])

shown in Fig. 3. In this case, the number of the smoothing domain is the same as the number of all the nodes in the Tr3-mesh. A smoothing domain forms a polygon bounded by multiple line segments. Any segment connects the midpoint of an edge to a center of a Tr3 element connected to the node. For the shaded node-based smoothing domain shown in Fig. 3, it is supported by 5 elements and it is thus bounded by 10 line segments. The number of supporting nodes of the interior edge-based smoothing domain k is 6, and they are A, B, C, D, E , and k . A node-based smoothing domain for a node on the problem boundary can be one-sided, and the number of supporting nodes is usually

smaller. In general a node-based smoothing domain may have contributions from any number of elements, as shown in Fig. 3. Mostly, one to seven elements.

For 3D problems, one may create edge-based S-FEM using 4-noded tetrahedral mesh, known as ES-FEM-Te4, as shown in Fig. 4. The supporting nodes of an edge-based smoothing domain are all the nodes of the element directly connected to the edge. We can also create node-based smoothing domains for NS-FEM-Te4 models, as shown in Fig. 5. In this case, the supporting nodes of a node-based smoothing domain are all the nodes of the element directly connected to the node. In addition, one may create

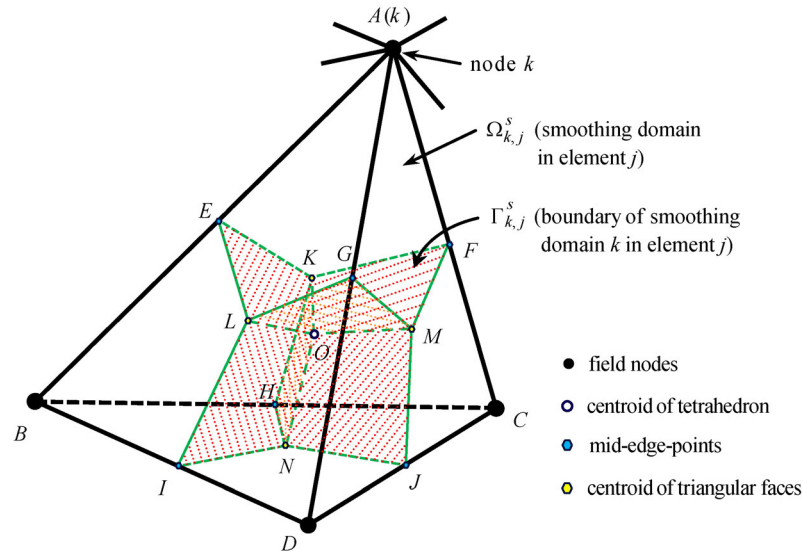


Fig. 5 Node-based smoothing domain on a part of Te4-mesh. Only part of the smoothing domain $\Omega_{k,j}^s$ is shown. It is for node k in the element j , and it is a polyhedron $AELGMFKO$. If there are other elements connected to node k , similar partial smoothing domains need to be constructed (from Ref. [167])

smoothing domains associated with the faces of the Te4 elements, known as the FS-FEM-Te4 model. Figure 6 shows a part of a face-based smoothing domain Ω_k^s created by connecting the three nodes of face (A,B,C) to the centers of these two neighboring elements (P,Q) . The supporting nodes of a face-based smoothing domain are all the nodes of the element directly connected to the face. For an interior face, the number of supporting nodes is 5, and for a face on the problem boundary, the number is 4, because only one Te4 element is involved in that smoothing domain.

Table 1 summaries several types of smoothing domains. More detailed procedure for constructing the smoothing domains can be found in Ref. [8].

Figures 7 and 8 show types of smoothing domains created on a 3D mechanical component (an engine

connection bar and socket, respectively) discretized with 4-noded tetrahedral elements, together with some solutions obtained using S-FEM models. Details on the formulation of an S-FEM model are given in the next Section.

2.3 S-FEM strain matrix (B-matrix)

We are now ready to form the strain-displacement matrix or B-matrix. For better clarity, we use 2D problem as an example. We write the assumed displacement $\mathbf{u}^h(\mathbf{x})$ in shape functions $\mathbf{N}_I(\mathbf{x})$ that satisfies the most essential condition of partitions of unity [199] and the nodal displacements \mathbf{d}_I for all the nodes of the elements contributing to the smoothing domain (SD):

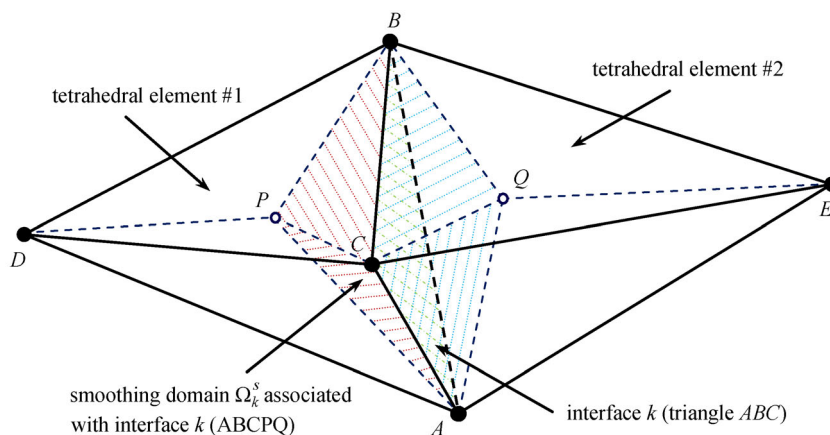


Fig. 6 Schematic illustration of face-based smoothing domains based on tetrahedral elements: a face-based smoothing domain Ω_k^s created from two adjacent tetrahedral elements based upon their interface k (from Ref. [167])

Table 1 Existing types of smoothing domains (SD's) used in S-FEM models

Type*	method for creation and number of SD's (N_s)	S-FEM models	dimension of problem; properties
cell-based SD (CSD)	SD's or smoothing cells (SC's) are divided from and located within the elements ($N_s = \sum_{i=1}^{N_e} n_{sc}^i$, $n_{sc}^i = 1, 2, 3, 4, \dots$)	CS-FEM n CS-FEM	1D, 2D, 3D softer; high accuracy; insensitive to mesh distortion
edge-based SD (ESD)	SD's are created based on edges by connecting portions of the surrounding elements sharing the associated edge ($N_s = N_{edge}$)	ES-FEM	2D, 3D softer; very high accuracy; less insensitive to mesh distortion
node-based SD (NSD)	SD's are created based on nodes by connecting portions of the surrounding elements sharing the associated node ($N_s = N_{node}$)	NS-FEM	1D, 2D, 3D soft; upper bound; very insensitive to mesh distortion; volumetric locking free
face-based SD (FSD)	SD's are created based on faces by connecting portions of the surrounding elements sharing the associated face ($N_s = N_{face}$)	FS-FEM	3D softer; very high accuracy; less insensitive to mesh distortion

*Note: There are S-FEM models that use combinations of different types of SDs, such as selective S-FEM, α FEM, and β FEM.

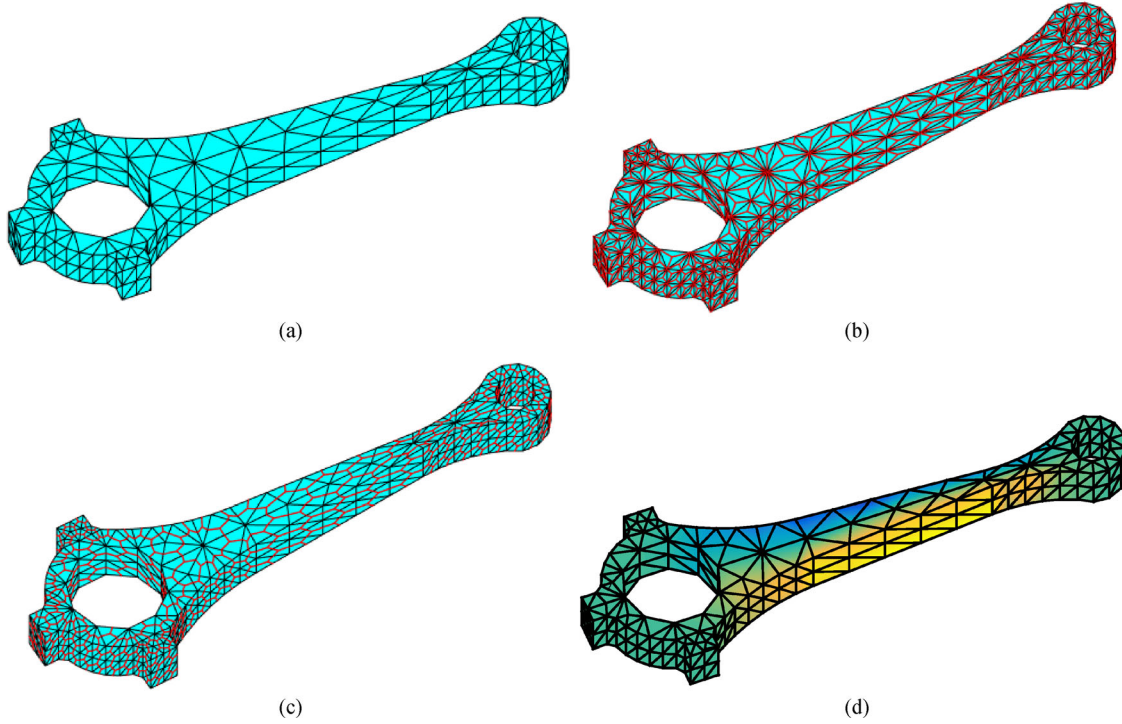


Fig. 7 Types of smoothing domains created on a 3D mechanical component (engine connection bar) discretized with 4-noded tetrahedral elements. (a) Face-based smoothing domains (on the surface the FS smoothing domains cannot be seen, and hence it appears like the element mesh); (b) edge-based smoothing domains; (c) node-based smoothing domains; (d) an example of a normal stress σ_{xx} solution using the ES-FEM-Tet4 model [163]

$$\bar{\mathbf{e}}_k = \sum_{I \in S_k^n} \bar{\mathbf{B}}_{Ik} \mathbf{d}_{Ik}, \quad (8)$$

where S_k^n is the set of supporting nodes of Ω_k^s , and the smoothed B-matrix can be computed by

$$\bar{\mathbf{B}}_{Ik} = \frac{1}{V_k^s} \int_{\Gamma_k^s} \mathbf{n}^s(\mathbf{x}) \mathbf{N}_I(\mathbf{x}) d\Gamma = \begin{bmatrix} \bar{b}_{Ikx} & 0 & \bar{b}_{Iky} \\ 0 & \bar{b}_{Iky} & \bar{b}_{Ikx} \end{bmatrix}^T, \quad (9)$$

where V_k^s is the area of the k th smoothing domain.

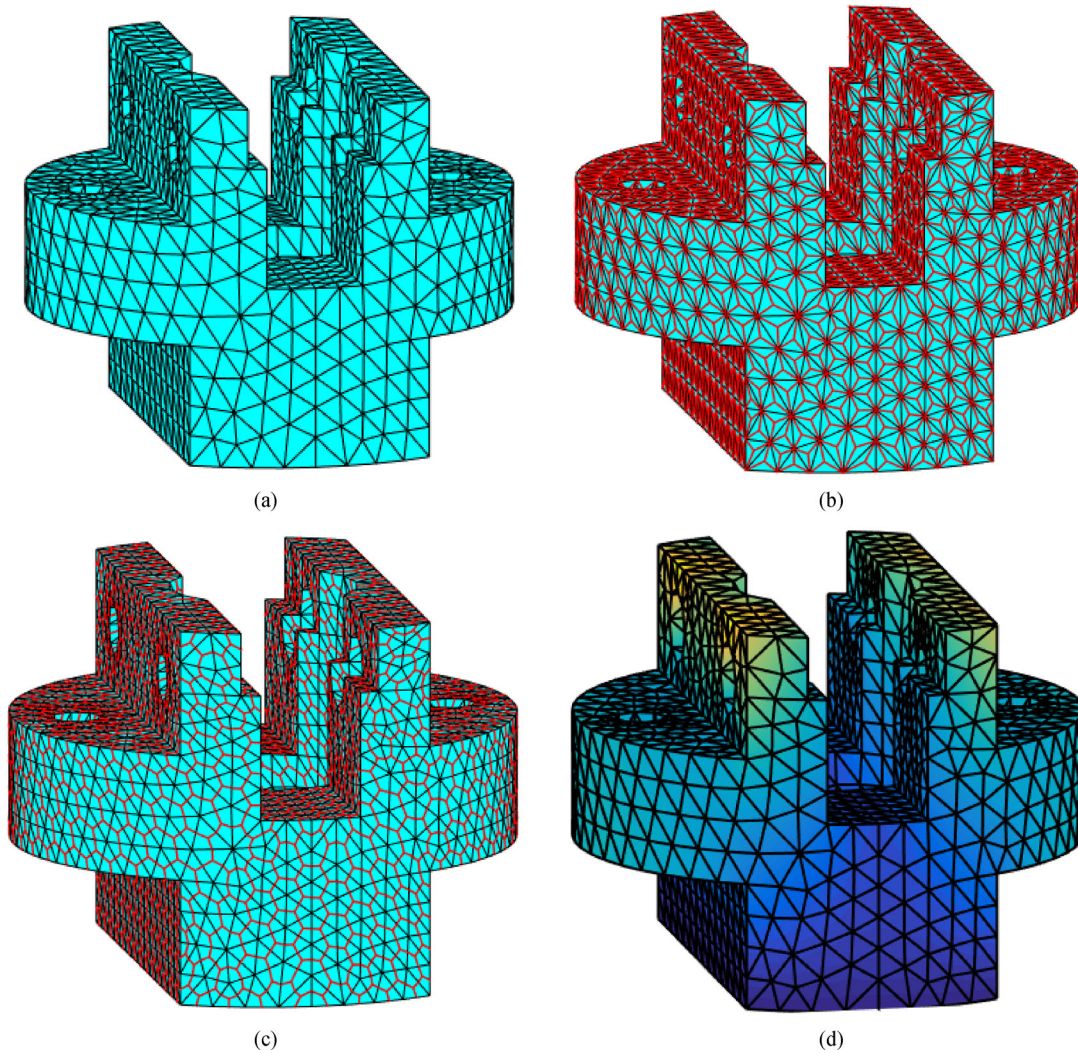


Fig. 8 Types of smoothing domains created on a 3D mechanical component (socket) discretized with 4-noded tetrahedral elements. (a) Face-based smoothing domains (on the surface the FS smoothing domains cannot be seen, and hence it appears like the element mesh); (b) edge-based smoothing domains; (c) node-based smoothing domains; (d) an example of a solution of displacement in the z-direction using the FS-FEM-Tet4 model [163]

$$\begin{aligned}\bar{b}_{lkh} &= \frac{1}{V_k^s} \int_{\Gamma_k^s} N_l(\mathbf{x}) n_h^s(\mathbf{x}) d\Gamma \\ &= \frac{1}{V_k^s} \sum_{p=1}^{n_r^s} n_{h,p}^s N_l(\mathbf{x}_p^G) l_p^s, \quad h = x, y, \quad (10)\end{aligned}$$

in which n_r^s is the total number of the boundary segments $\Gamma_{k,p}^s \in \Gamma_k^s$, \mathbf{x}_p^G is the Gauss point on $\Gamma_{k,p}^s$. The length and outward unit normal of $\Gamma_{k,p}^s$ are denoted as l_p^s and $n_{h,p}^s$, respectively. Here we use 1-point Gauss quadrature to numerically perform the integration along each segment. It is clearly seen in Eq. (10) that in computing B-matrix, derivatives of shape functions N_l are not required, and we need only the shape functions values and only for the points on the SD boundary segments. Once the SDs are

created, one knows exactly the supporting elements and nodes for each SD, and hence the shape function values for all the supporting nodes at any point on the SD boundary segments can be easily computed using the PIM [6,8]. The compatibility is not a concern at all for an S-FEM model. This is because the PIM is performed for points on the SD boundary, it satisfies the continuity requirement for any function being in a G_h^1 space, and hence the stability is ensured [5,6].

For the case of using 4 SDs in a Q4 element (shown in Fig. 9), the values of the 4 nodal shape functions at the 12 Gauss points on these boundary segments of the 4 SDs can be computed by simple PIM (in fact a simple inspection), as listed in the Table 3 [8]. The key point here is that we do not need to construct these shape functions of each of these 4 nodes (this would be, in fact, a nontrivial task, because it

Table 2 Minimum number of smoothing domains N_s^{\min} for solid mechanics problems with n_t (unconstrained) total nodes [5,8]

dimension of the problem	minimum number of smoothing domains
1D	$N_s^{\min} = n_t$
2D	$N_s^{\min} = 2n_t/3$
3D	$N_s^{\min} = 3n_t/6 = n_t/2$

needs on in the physical coordinate system, and hence mapping is required as in the FEM). Note also that mapping is not required in S-FEM. Therefore, it is more computationally efficient and simpler in this regards. Such a PIM can also be used for any arbitrary polygonal elements, as the 6-sided polygonal element shown in Fig. 9 (a), and the results for all these Gauss points are listed in Table 4.

Note that the summation in Eq. (8) is in fact an “assembly” or a “node-matched” summation. As an example, let us consider an ES-FEM model for 2D problems. In this case, the (shaded) edge-based SD $DPFQ$ shown in Fig. 2 is supported by 4 nodes D, E, F, G from two Tr3 elements DEF and DFG . The smoothed B-matrix for the whole SD Ω_k^s can then be written as Ref. [8]

$$\bar{\mathbf{B}}_k = [\bar{\mathbf{B}}_{Dk} \bar{\mathbf{B}}_{Ek} \bar{\mathbf{B}}_{Fk} \bar{\mathbf{B}}_{Gk}]. \quad (11)$$

All there sub-matrices in the right-hand-side of the forgoing equation can be computed easily using Eqs. (9) and (10). For the ES-FEM-T3 model, only one Gauss point is required along any boundary segment, because the shape function changes linearly and the unit normal vector is a constant along the segments.

For Tr3 elements, the area of a SD can be calculated using the areas of the elements supporting the SD:

$$V_k^s = \int_{\Omega_k^s} d\Omega = \frac{1}{3} \sum_{j=1}^{n_k^e} V_j^e, \quad (12)$$

where n_k^e is the number of elements connected to edge k and V_j^e is the area of an element.

The above is the standard way to compute the smoothed B-matrix. Alternatively, one may use the area-weighted summation method, if Tr3 elements and linear PIM is used. In this method, $\bar{\mathbf{B}}_{lk}$ is computed using directly all the compatible FE \mathbf{B}_I^{e-j} for the j th element that is connected to edge k . \mathbf{B}_I^{e-j} for node I can be evaluated using the shape function of node I in the j th element:

Table 3 Values of 4 nodal shape functions at different points within a Q4 element [8] (shown in Fig. 9(a))

point	N_1	N_2	N_3	N_4	description
1	1.0	0	0	0	field node
2	0	1.0	0	0	field node
3	0	0	1.0	0	field node
4	0	0	0	1.0	field node
5	1/2	1/2	0	0	side midpoint
6	0	1/2	1/2	0	side midpoint
7	0	0	1/2	1/2	side midpoint
8	1/2	0	0	1/2	side midpoint
9	1/4	1/4	1/4	1/4	intersection of two bi-medians
g1	3/4	1/4	0	0	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g2	3/8	3/8	1/8	1/8	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g3	3/8	1/8	1/8	3/8	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g4	3/4	0	0	1/4	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g5	1/4	3/4	0	0	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g6	0	3/4	1/4	0	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g7	1/8	3/8	3/8	1/8	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g8	0	1/4	3/4	0	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g9	0	0	3/4	1/4	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g10	1/8	1/8	3/8	3/8	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g11	0	0	1/4	3/4	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)
g12	1/4	0	0	3/4	Gauss point (mid-segment point of $\Gamma_{k,p}^x$)

Table 4 Values of six nodal shape functions at different points within a 6-sided polygonal element [8] (shown in Fig. 9(b))

point	N_1	N_2	N_3	N_4	N_5	N_6	description
1'	1.0	0	0	0	0	0	field node
2'	0	1.0	0	0	0	0	field node
3'	0	0	1.0	0	0	0	field node
4'	0	0	0	1.0	0	0	field node
5'	0	0	0	0	1.0	0	field node
6'	0	0	0	0	0	1.0	field node
O	1/6	1/6	1/6	1/6	1/6	1/6	centroid point
g1'	7/12	1/12	1/12	1/12	1/12	1/12	Gauss point (mid-segment point of Γ_{kp}^s)
g2'	1/2	1/2	0	0	0	0	Gauss point (mid-segment point of Γ_{kp}^s)
g3'	1/12	7/12	1/12	1/12	1/12	1/12	Gauss point (mid-segment point of Γ_{kp}^s)
g4'	0	1/2	1/2	0	0	0	Gauss point (mid-segment point of Γ_{kp}^s)
g5'	1/12	1/12	7/12	1/12	1/12	1/12	Gauss point (mid-segment point of Γ_{kp}^s)
g6'	0	0	1/2	1/2	0	0	Gauss point (mid-segment point of Γ_{kp}^s)
g7'	1/12	1/12	1/12	7/12	1/12	1/12	Gauss point (mid-segment point of Γ_{kp}^s)
g8'	0	0	0	1/2	1/2	0	Gauss point (mid-segment point of Γ_{kp}^s)
g9'	1/12	1/12	1/12	1/12	7/12	1/12	Gauss point (mid-segment point of Γ_{kp}^s)
g10'	0	0	0	0	1/2	1/2	Gauss point (mid-segment point of Γ_{kp}^s)
g11'	1/12	1/12	1/12	1/12	1/12	7/12	Gauss point (mid-segment point of Γ_{kp}^s)
g12'	1/2	0	0	0	0	1/2	Gauss point (mid-segment point of Γ_{kp}^s)

$$\mathbf{B}_I^{e,j} = \mathbf{L}_d \mathbf{N}_I(\mathbf{x}). \quad (13)$$

The sub-smoothed B-matrix for node I is then computed using

$$\bar{\mathbf{B}}_{Ik} = \frac{1}{V_k^s} \sum_{j=1}^{n_k^e} \left[\frac{1}{3} V_j^e \mathbf{B}_I^{e,j} \right]. \quad (14)$$

For the example in Fig. 2, elements DEF and DFG

support the red shaded SD Ω_k^s . However, element DFG is not related to node E . When $\bar{\mathbf{B}}_{Ek}$ is computed, it has only 1/3 contribution of $\mathbf{B}_E^{e,DEF}$ from element DEF . Likewise, when $\bar{\mathbf{B}}_{Gk}$ is computed, 1/3 of $\mathbf{B}_G^{e,DEG}$ is contributed from DFG . However, when $\bar{\mathbf{B}}_{Dk}$ or $\bar{\mathbf{B}}_{Fk}$ is computed, we have contributions from both elements, as they all share nodes D and F . The smoothed B-matrix for the whole SD Ω_k^s can be written in the following form.

$$\bar{\mathbf{B}}_k = \left[\underbrace{\frac{1}{3} \mathbf{B}_D^{e,DEF} + \frac{1}{3} \mathbf{B}_D^{e,DEG}}_{\bar{\mathbf{B}}_{Dk}} \underbrace{\frac{1}{3} \mathbf{B}_E^{e,DEF}}_{\bar{\mathbf{B}}_{Ek}} \underbrace{\frac{1}{3} \mathbf{B}_F^{e,DEF} + \frac{1}{3} \mathbf{B}_F^{e,DEG}}_{\bar{\mathbf{B}}_{Fk}} \underbrace{\frac{1}{3} \mathbf{B}_G^{e,DEG}}_{\bar{\mathbf{B}}_{Gk}} \right]. \quad (15)$$

We noted that Eqs. (11) and (15) are identical, if Tr3 elements (linear PIM) are used. Equation (11) is standard and applicable to other types of elements and higher order PIMs (with of course more Gauss points for the integrations).

2.4 S-FEM stiffness matrix

The computation and formation of the smoothed stiffness matrix $\bar{\mathbf{K}}$ is quite similar to those procedures in the

standard FEM. It can be assembled from the contributions of the sub-stiffness-matrices from all the smoothing domains,

$$\begin{aligned} \bar{\mathbf{K}}_{IJ} &= \int_{\Omega} \bar{\mathbf{B}}_I^T \mathbf{c} \bar{\mathbf{B}}_J d\Omega = \sum_{k=1}^{N_s} \left[\int_{\Omega_k^s} \bar{\mathbf{B}}_{Ik}^T \mathbf{c} \bar{\mathbf{B}}_{Jk} d\Omega \right] \\ &= \sum_{k=1}^{N_s} \underbrace{\bar{\mathbf{B}}_{Ik}^T \mathbf{c} \bar{\mathbf{B}}_{Jk}}_{\mathbf{K}_{Ijk}^s} V_k^s, \end{aligned} \quad (16)$$

The summation is a node-matched summation at the stiffness matrix level. The derivation of the above equation is similar as that in the FEM. The main difference is that FEM is *element* based, while the S-FEM is *smoothing-domain* based. The existing assembly algorithms in the FEM can be used for S-FEM by simply treating the smoothing domains as “elements”. When I and J are “far” apart, $\bar{\mathbf{K}}_{IJ}$ vanishes. Thus, the global stiffness matrix $\bar{\mathbf{K}}$ is a sparse (assuming it is formed). It is banded when the nodes are properly numbered.

2.5 S-FEM discretized system equations

Consider now dynamic problems for solids and structures, the discretized system of equations in an S-FEM can be expressed as the following set of 2nd order differential equations with respect to time.

$$\bar{\mathbf{K}}\mathbf{d} + \tilde{\mathbf{C}}\dot{\mathbf{d}} + \tilde{\mathbf{M}}\ddot{\mathbf{d}} = \tilde{\mathbf{f}}, \quad (17)$$

where $\tilde{\mathbf{M}}$ is the mass matrix obtained using

$$\tilde{\mathbf{M}} = \int_{\Omega} \mathbf{N}^T \rho \mathbf{N} d\Omega, \quad (18)$$

in which ρ is the mass density, and \mathbf{N} is the matrix of nodal shape functions of all nodes [4,8]. Matrix $\tilde{\mathbf{C}}$ is the damping matrix computed using

$$\tilde{\mathbf{C}} = \int_{\Omega} \mathbf{N}^T c_d \mathbf{N} d\Omega, \quad (19)$$

where c_d is the damping coefficient of the material. Vector $\tilde{\mathbf{f}}$ is the external force vector acting at all the nodes in the problem domain. It has entries of

$$\tilde{\mathbf{f}}_I = \int_{\Omega} \mathbf{N}_I^T(\mathbf{x}) \mathbf{b} d\Omega + \int_{\Gamma_f} \mathbf{N}_I^T(\mathbf{x}) \mathbf{t} d\Gamma, \quad (20)$$

where \mathbf{b} is the distributed body force vector, and the \mathbf{t} is the traction vector applied on the force boundary of the problem domain.

In the S-FEM, the smoothing operation is only applied to the derivatives of the displacement (or shape) functions. We do not perform any additional treatments to the displacement function itself. Therefore, the mass matrix, damping matrix, and the force vectors are computed in exactly the same way as in the standard FEM. The damping matrix may also be modeled as in the standard FEM. For example using the so-called Rayleigh damping. In such a case the damping matrix $\tilde{\mathbf{C}}$ is assumed to be a linear combination of $\tilde{\mathbf{M}}$ and $\bar{\mathbf{K}}$,

$$\tilde{\mathbf{C}} = \alpha \tilde{\mathbf{M}} + \beta \bar{\mathbf{K}}, \quad (21)$$

where α and β are the Rayleigh damping coefficients determined by experiments.

The stiffness matrix $\bar{\mathbf{K}}$ is a symmetric positive definite (SPD), after sufficient displacement boundary conditions are imposed [4,8], as long as the number of the smoothing domains satisfy the following table of stability conditions.

The bandwidth of $\bar{\mathbf{K}}$ depends on the types of S-FEM model. For CS-FEM-Q4, it is the same as the FEM. For ES-FEM-Tr3, the bandwidth is about 30% larger than the FEM counterpart. For NS-FEM-Tr3, the bandwidth is doubled. Therefore, if direct solver is used for Eq. (17), NS-FEM is expected slower than ES-FEM that is also slower than the FEM counterpart using the same mesh. The S-FEM can, however, stand out by producing more accurate solutions and/or offering unique solution properties.

Note that when explicit solver is used for dynamic problems, we do not need to form matrix $\bar{\mathbf{K}}$ during the computation. In such cases, the computation time is largely determined by the number of smoothing domains (elements in the case of FEM). In this case, NS-FEM can be faster than ES-FEM that is also faster than the FEM counterpart using the same mesh. This is because for a mesh the number of nodes is usually smaller than the number of elements and that even smaller than the number of edges. The S-FEM can also stand out further by producing more accurate solutions or offer unique solution properties.

3 Solution properties of S-FEM models

3.1 Example 1: 2D cantilever beam

Next we study a benchmarking mechanics problem known as 2D cantilever beam, which has an analytical solution given in [8,191]. The beam is a simple rectangular shape with a length $L = 48$ m and height $D = 12$ m is subjected to a parabolic traction at the free end as in Fig. 10. When we assume the thickness is very small comparing with its height, and thus it is considered as a 2D plane-stress problem. Because we have exact solution, we can use it to examine our numerical models in detail.

Convergence of numerical solution in the strain energy are obtained using various numerical methods for this 2D cantilever problem, and the results are plotted in Fig. 11. A set of uniformly distributed 3-noded triangular elements are used to discretize the problem domain, and the density of the mesh is controlled by the DoFs. It is found that the FEM solution produces a lower bound, the NS-FEM gives an upper bound, and the ES-FEM gives ultra-accurate solution. All these numerical models use exactly the same element mesh, but different types of smoothing domains (note that the FEM is in fact the same as CS-FEM-Tr3). The findings from this example are in-line with the predictions by our S-FEM theory. This demonstrates an important point that we can now design numerical models

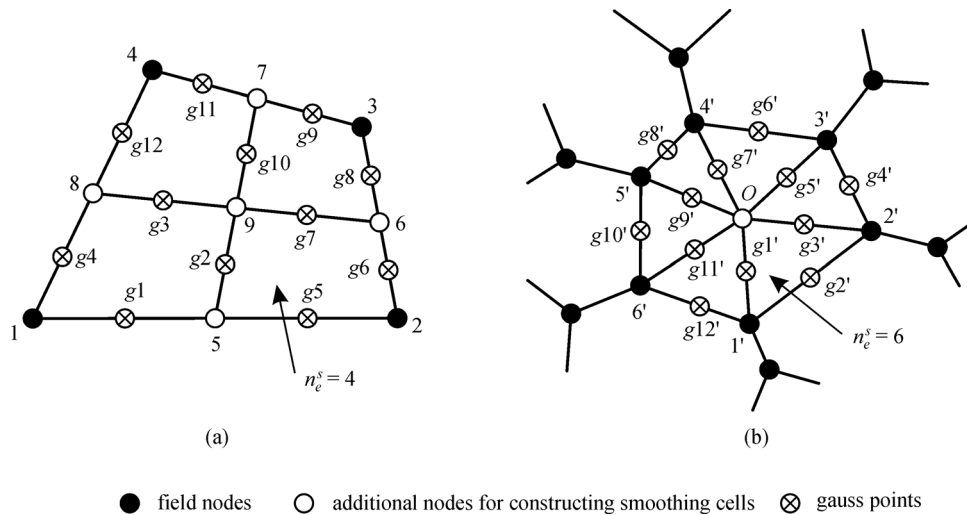


Fig. 9 Positions of Gauss points at mid-segment-points on segments of smoothing domains. (a) Four quadrilateral smoothing domains in a Q4 element; (b) six triangular smoothing domains in a 6-sided polygonal element (from Ref. [8])

with different properties by simply using different types of smoothing domains.

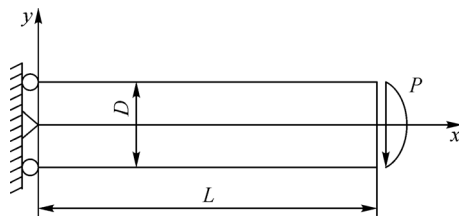


Fig. 10 A 2D cantilever beam loaded by a downward parabolically distributed shear stress at the right end

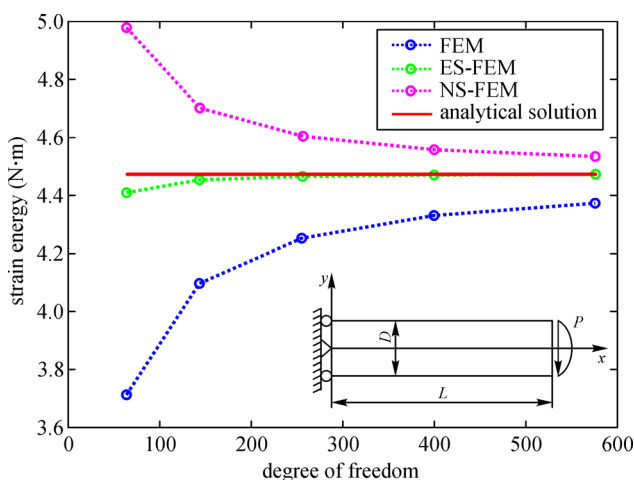


Fig. 11 Convergence of numerical solution in the strain energy for the 2D cantilever problem (from Ref. [163])

3.2 Example 2: 3D cantilever cubic solid

We next consider a 3D mechanics problem of cantilever

cubic solid. Its three dimensions are $L = W = H = 1$ m, and it is subjected to a uniform pressure loading on the upper face shown in Fig. 12. The cubic solid is fixed on its left face. Young's modulus of the solid material is $E = 1.0 \times 10^3$ N/m² and Poisson's ratio is $\nu = 0.3$. This problem seems simple, but there is no exact solution to it. To conduct a detailed analysis for our numerical methods, we need to use a reference solution. Almeida Pereira has provided such a solution using a solution obtained with a very fine mesh of hexahedral super-elements, together with a procedure of Richardson's extrapolation [191]. Such a reference solution is a good approximation of the exact solution and are used by many for examining numerical models. The solution in strain energy is 0.95093.

Figure 13 plots the convergence curves of numerical solutions obtained using different numerical models in

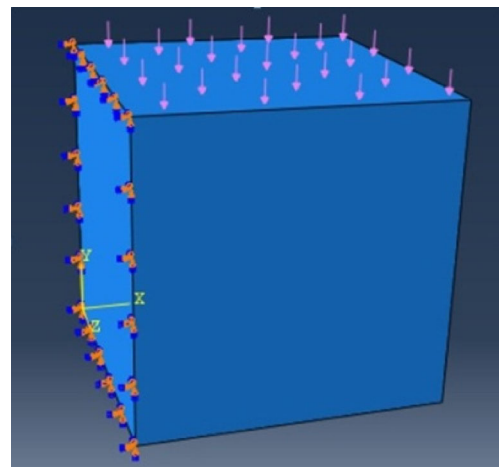


Fig. 12 A 3D cantilever cubic solid fixed on its left face, and it is subjected to a uniformly distributed pressure loading on the top surface

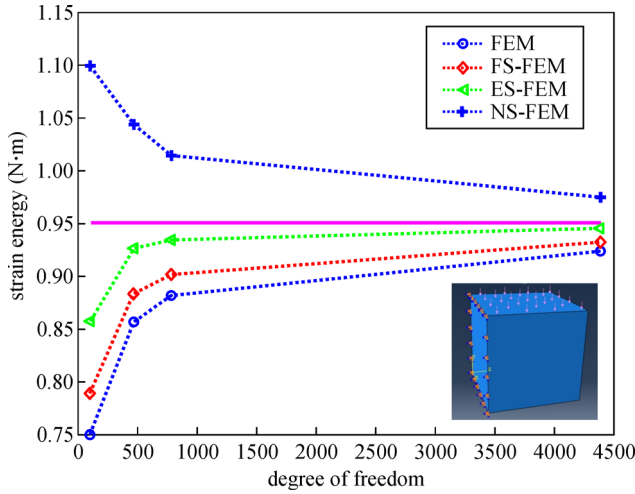


Fig. 13 Convergence of numerical solution in strain energy for the 3D cantilever cubic solid (from Ref. [163])

strain energy for the 3D cantilever cubic solid. A set of uniformly distributed 4-noded tetrahedral elements are used to discretize the problem domain, and the density of the mesh is controlled by the DoFs. It is found again that the FEM solution is a lower bound, the NS-FEM gives an upper bound, and the ES-FEM gives ultra-accurate solution. The solution from the FS-FEM is also much more accurate than the FEM counterpart. The all use exactly the same element mesh, but different types of smoothing domain. The findings are again in-line with the predictions by the S-FEM theory. This demonstrates again that we can now design numerical models with different properties by simply using different types of smoothing domains.

Figure 14 shown an important and useful idea based on the properties of S-FEM models. In the S-FEM framework, we have now two knobs: one tunes the stiffening effects by properly assuming the displacement field for the model, and another tunes the softening effects by strain smoothing operations (via using different types of smoothing domains). To obtain an lower bound solution, we tun up

the left knob, and to obtained a upper bound solution, we tun the right knob. In theory, one can produce an S-FEM model by design that can produce exact solution at least in a norm for a mechanics problem for solids and structures.

4 Moving forward

The development of S-FEM has already opened a new window of opportunity to develop the next generation of computational methods. Moving forward, it is the author's opinion that the S-FEM will advances fast in the following areas:

1) Development of commercial software packages using S-FEM technology. Because S-FEM works well with T-mesh, we need now only use T-mesh that can be automatically generated for complex geometry. This is also extremely important for our dream for fully-automation in computation, modeling, and simulation. The manual operations from the analyst of a project will be drastically reduced. S-FEM software and the pre-processes can also be much simpler compared to the FEM counterpart, because of the use of simplest T-mesh. Some of the basic S-FEM codes for various models are available for free download (a simple registration for records is needed) at GRLab's website, which offers a good initial starting point.

2) The automation capability of S-FEM offers conveniences in creating real-time AI models for mechanics problems, based on neural networks [207,208]. The AI methods are basically data-based, and the current biggest bottleneck problem with AI is the difficulty to obtain a large number of training samples. The S-FEM is physics-based, and it can be used for generating training samples for neural networks, by creating automatically S-FEM models using T-meshes. Because manual operations are drastically reduced for model creation, one can create as many training samples as needed. This is particularly important for inverse problems [209].

3) A lot more intensive applications of S-FEM is expected in practical applications in sciences and engi-

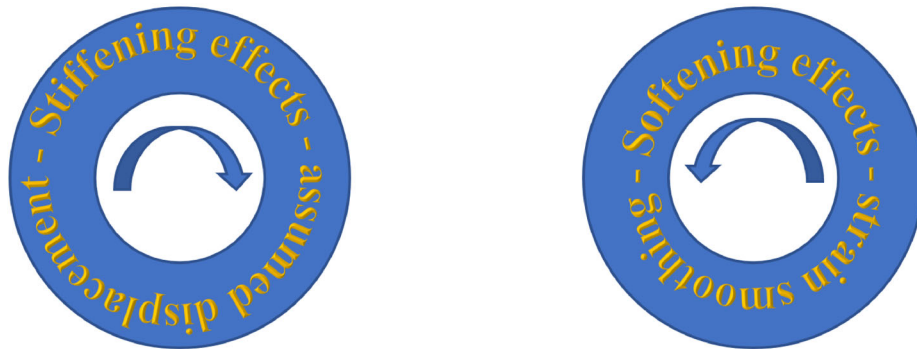


Fig. 14 In the S-FEM framework, we have now two knobs: one on the left tunes the stiffening effects by properly assuming the displacement field for the model, and another on the right tunes the softening effects by strain smoothing operations

neering fields, especially problems require adaptive analyses. Even more innovative ways to construct new types of smoothing domains. So far the smoothing domains in the existing S-FEM models are created in tight relations with the element mesh. This is not necessary. In theory, the smoothing domains can be independent of element mesh. A recent work by Liu's group has made some initial attempt in this direction [210].

4) Higher order S-FEM models. The S-FEM models developed so far are mainly linear models (linear PIM shape functions, and SD-based piecewise linear strain field). Such linear S-FEM models shall suffice for most of the applications (based on our experiences in using FEM, linear and bilinear models are most widely used, even though higher FEM elements are available in most software packages). However, higher order models will be an important addition. Liu has recently developed a *pick-out theory* and a systematic approach to construct higher order smoothed strain field [211]. The development of higher order S-FEM models is already on the way. The use of robust radial basis functions can also be a new direction of development for more robust and higher order formulations [194,200].

5) Most importantly we may need to develop ideas to make full use of the S-FEM frame work that enables us to develop models in convenient manner based the demand of the analyst on the required solution property. This requires a change on the perception of a numerical model.

5 Concluding remarks

This article first provides a brief review on the widely used FEM, and then a concise and easy-to-following presentation of key minimum necessary formulae used in the S-FEM. Following this concise introduction, reader shall be able to understand the essence of S-FEM and code S-FEM models based on the basic codes provided at GRLab's website. We provided also new directions on further development of S-FEM technology.

A number of important properties and unique features of S-FEM models are discussed in detail, which helps readers to understand better and appreciate the method. Most importantly, a concept of numerical model on-demand based on the S-FEM framework is proposed that may drastically change the landscape of modeling and simulation. One can in fact purposely design an S-FEM model to obtain solutions with special properties. This changes the perception of a numerical model. We used to treat a numerical model as merely tool for analysis to optimize our product design. With the S-FEM framework, we can now have a means to optimize the tool itself for desired solutions, which can next be utilized for much more reliable analyses and then to optimize our product design with high confidence. For example, for an automatically

generated T-mesh, one can create automatically different types of smoothing domains [163,190]. By invoking NS-FEM one gets upper bound solution, FEM for lower bound solution, and ES-FEM for solutions with high accuracy. This new concept of numerical model On-Demand is valuable for full automation in computations and adaptive analyses, and hence has profound impact on the future AI-assisted modeling and simulation.

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