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REVIEW ARTICLE

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An overview on the applications of the hesitant fuzzy sets in group decision-making: theory, support and methods

Abstract Due to the characteristics of hesitant fuzzy sets (HFSs), one hesitant fuzzy element (HFE), which is the basic component of HFSs, can express the evaluation values of multiple decision makers (DMs) on the same alternative under a certain attribute. Thus, the HFS has its unique advantages in group decision making (GDM). Based on which, many scholars have conducted in-depth research on the applications of HFSs in GDM. We have viewed lots of relevant literature and divided the existing studies into three categories: theory, support and methods. In this paper, we elaborate on hesitant fuzzy GDM from these three aspects. The first aspect is mainly about the introduction of HFSs, HFPRs and some hesitant fuzzy aggregation operators. The second aspect describes the consensus process under hesitant fuzzy environment, which is an important support for a complete decision-making process. In the third aspect, we introduce seven hesitant fuzzy GDM approaches, which can be applied in GDM under different decision-making conditions. Finally, we summarize the research status of hesitant fuzzy GDM and put forward some directions of future research.

Keywords hesitant fuzzy set, hesitant fuzzy preference relation, group decision-making

1 Introduction

GDM is an interdisciplinary subject integrating mathematics, economics, social psychology, behavioral science, management science and many other disciplines. An important characteristic of GDM problems is multiple complexities: (1) There often exist several conflicting goals in GDM problems; (2) DMs often have different preferences for attributes and alternatives. Thus, in the GDM process, contradictions and conflicts become inevitable problems. Therefore, the GDM process is usually a process of interactively seeking satisfactory decision-making results.

Due to the complexity of attributes, the members of decision-making groups often have different social, economic, cultural backgrounds, and have specific knowledge structures and behavioral characteristics. These will be reflected in the DMs' preferences for the alternatives. Traditional decision-making methods usually assume that the DMs are completely rational, that is to say, the decision-making behaviors of the DMs must be based on the completeness of knowledge and the consistency of values or preferences. This is definitely impossible. In practical decision-making processes, due to inherent physiological conditions and different educational backgrounds, the DMs cannot have complete knowledge and consistent values or preferences. That is to say, the DMs are always boundedly rational. They can only give choices or assessments that satisfy themselves, but not the same as others, not the optimal solution.

The above facts tell us that the GDM problems often have strong uncertainty, which is represented as fuzziness within the decision-making group. For example, the limitations of the knowledge of some experts and the different preferences among different experts will make the evaluation information given by the decision-making group a certain degree of fuzziness. Therefore, in today's environment, the GDM methods pay more and more attention to the fuzzy information contained in the GDM problems.

Specifically, when an expert gives an evaluation value for a program, he/she is likely to hesitate in a few values. Moreover, when more than one expert evaluates an alternative, it is very possible for them to have different opinions. This is not because of the error in the initial decision-making information, but because those evaluation values should be represented by a set of several possible values.

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In 2010, Torra (2010) proposed the concept of HFS. The basic components of HFSs are HFEs, which is consisted of several possible values between 0 and 1. Therefore, compared to other extension forms of fuzzy sets, the HFS can describe the hesitant information of the DMs more comprehensively and meticulously. Then, Xia and Xu (2011) gave the mathematical expression of HFS and defined the concept of HFE. Since then, HFS theory has developed rapidly and has been widely used in various decision-making processes, including the GDM field. In fact, the characteristics of HFEs are very compatible with the GDM problems. For example, when the evaluation values given by experts are different, we can integrate them into an HFE. Thus, several isolated evaluation values become one with the HFE form. This makes many single decision-making approaches be applied to GDM (especially multi-attribute group decision-making (MAGDM)) problems. In this way, the calculation processes of lots of GDM problems are effectively simplified.

We searched the Internet for a large number of documents on the scope of applications of HFS in GDM and selected dozens of them which are more important. It can be clearly seen that the scope of applications of HFS in GDM is mainly divided into the following three parts:

(1) The hesitant fuzzy sets and hesitant fuzzy preference relations (HFPRs);

(2) A decision support for GDM — the consensus processes under hesitant fuzzy environments;

(3) The hesitant fuzzy MAGDM methods.

This paper summarizes the applications of HFEs in GDM from the above three perspectives. In Section 2, we introduce the concepts, properties, some operations and typical aggregation operators of HFSs (HFEs) and three kinds of HFPRs. Section 3 introduces the consensus process under hesitant fuzzy environment, which provides an important support for the solutions of hesitant fuzzy GDM problems. Section 4 introduces several hesitant fuzzy MAGDM methods, which provide some practical ways to solve the hesitant fuzzy MAGDM problems under different conditions. The above three sections are the main lines of the paper, which basically cover the whole process of solving a hesitant fuzzy GDM problem. Moreover, the methods in them are also more fundamental and commonly used. The last section supplements the contents of the previous sections and introduces some scattered methods for solving hesitant fuzzy GDM problems. Besides, we put forward some directions of the applications of HFS theory in GDM.

2 Hesitant fuzzy aggregation operators and preference relations

As mentioned earlier, HFSs can be applied to many decision-making circumstances, especially the MAGDM problems. In order to get the best alternative in a MAGDM problem, the scholars have proposed two common ways (Xia and Xu, 2011b): (1) Integrate all the DMs' opinions on a certain alternative for each attribute, and then integrate the evaluation values of all attributes of the alternative; (2) For each alternative, we integrate the evaluation values of all attributes given by a certain DM, and then integrate the opinions of all DMs with respect to each alternative. Regardless of the method, hesitant fuzzy aggregation operators are an important tool. In this section, we introduce the concepts and some properties of HFSs and HFEs and the existing hesitant fuzzy aggregation operators.

2.1 Hesitant fuzzy elements

Definition 1 (Torra, 2010)**.** Let X be a fixed set, then, a hesitant fuzzy set on X is a function that maps each element of X to a subset of [0,1].

For easy understanding, Xia and Xu (2011b) gave the mathematical expression of the HFS as follows:

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ is a set of values in [0,1], which expresses the membership degrees to which an element x is attached to a set A. $h = h_A(x) = \{\gamma_1,\gamma_2,\ldots,\gamma_{|h|}\}$ is called the hesitant fuzzy element, where $\gamma_i (i = 1,2,\ldots,|h|)$ are the terms of h, and $|h|$ is the number of terms in h. We use Θ to express the set of all HFEs.

The following definition gives the concepts of score values and deviation degrees of HFEs.

Definition 2 (Xia and Xu, 2011b; Chen et al., 2013a)**.** Suppose that h is an HFE, then $s(h) = \frac{1}{|h|}\sum_{\gamma\in h}\gamma$ is called the score value of h. $\sigma(h) = \sqrt{\frac{1}{|h|}\sum_{\gamma\in h}\left(\gamma - s(h)\right)^2}$ is called the deviation degree of h.

Based on the above two concepts, a method to compare the sizes of two HFEs can be obtained:

Suppose that h_1 and h_2 are two HFEs, $s(h_1)$ and $s(h_2)$ are the score values of h_1 and h_2, respecively. $\sigma(h_1)$ and are the deviation degrees of h_1 and h_2, respectively. Then,

(1) If $s(h_1) < s(h_2)$, then $h_1 < h_2$;

(2) If $s(h_1) > s(h_2)$, then $h_1 > h_2$;

(3) If $s(h_1) = s(h_2)$, then

(i) If $\sigma(h_1) < \sigma(h_2)$, then $h_1 > h_2$;

(ii) If $\sigma(h_1) > \sigma(h_2)$, then $h_1 < h_2$;

(iii) If $\sigma(h_1) = \sigma(h_2)$, then $h_1 = h_2$.

Some operations and properties of HFEs can be introduced as follows:

Definition 3 (Torra, 2010; Xia and Xu, 2011b; Liao and Xu, 2014b)**.** Let h, h_1 and h_2 be three HFEs, $\lambda>0$ is a real number. Then,

(1) $h^c = \cup_{\gamma \in h}\{1-\gamma\}$;

(2) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$;

(3) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$;

(4) $h^\lambda = \cup_{\gamma \in h} \gamma^\lambda$;

(5) $\lambda h = \cup_{\gamma \in h}\{1-(1-\gamma)^\lambda\}$;

(6) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

(7) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$;

(8) $h_1 \ominus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\bar{\gamma}\}$,

where $\bar{\gamma} = \begin{cases} \dfrac{\gamma_1 - \gamma_2}{1-\gamma_2}, \gamma_1 \geqslant \gamma_2, \gamma_2 \neq 1 \\ 0, \text{ others} \end{cases}$;

(9) $h_1 \oslash h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\bar{\gamma}\}$,

where $\bar{\gamma} = \begin{cases} \dfrac{\gamma_1}{\gamma_2}, \gamma_1 \leqslant \gamma_2, \gamma_2 \neq 0 \\ 1, \text{ others} \end{cases}$.

Next, we briefly introduce some properties of HFEs.

Property 1 (Xia and Xu, 2011b; Zhu et al., 2012)**.** For any three HFEs h, h_1 and h_2, $\lambda > 0$, the following equations hold:

(1) $h_1^c \cup h_2^c = (h_1 \cap h_2)^c$,

$h_1^c \cap h_2^c = (h_1 \cup h_2)^c$;

(2) $(h^c)^\lambda = (\lambda h)^c$,

$\lambda (h)^c = (h^\lambda)^c$;

(3) $h_1^c \oplus h_2^c = (h_1 \otimes h_2)^c$,

$h_1^c \otimes h_2^c = (h_1 \oplus h_2)^c$;

(4) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$,

$(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda$;

(5) $(h_1 \oplus h_2) \oplus h_3 = h_1 \oplus (h_2 \oplus h_3)$,

$(h_1 \otimes h_2) \otimes h_3 = h_1 \otimes (h_2 \otimes h_3)$.

Property 2 (Liao and Xu, 2014b). For any four HFEs $h = \{\gamma^l | l = 1,2,...,|h|\}$, $h_1 = \{\gamma_1^i | i = 1,2,...,|h_1|\}$, $h_2 = \{\gamma_2^j | j = 1,2,...,|h_2|\}$ and $h_3 = \{\gamma_3^k | k = 1,2,...,|h_3|\}$, $\lambda, \lambda_1, \lambda_2 > 0$ and $\lambda_1 \geqslant \lambda_2$. Then, the following equations hold:

(1) $\lambda_1 h \ominus \lambda_2 h = (\lambda_1 - \lambda_2)h$,

when $\gamma^l \neq 1$,

$h^{\lambda_1} \oslash h^{\lambda_2} = h^{(\lambda_1 - \lambda_2)}$,

when $\gamma^l \neq 0$;

(2) $h_1^c \ominus h_2^c = (h_1 \oslash h_2)^c$,

$h_1^c \oslash h_2^c = (h_1 \ominus h_2)^c$;

(3) $(h_1 \ominus h_2) \oplus h_2 = h_1$,

when $\gamma_1^i \geqslant \gamma_2^j$,

and $\gamma_2^j \neq 1$,

$(h_1 \oslash h_2) \otimes h_2 = h_1$,

when $\gamma_1^i \leqslant \gamma_2^j$,

and $\gamma_2^j \neq 0$;

(4) $\lambda(h_1 \ominus h_2) = \lambda h_1 \ominus \lambda h_2$,

when $\gamma_1^i \geqslant \gamma_2^j$,

and $\gamma_2^j \neq 1$,

$(h_1 \oslash h_2)^\lambda = h_1^\lambda \oslash h_2^\lambda$,

when $\gamma_1^i \leqslant \gamma_2^j$,

and $\gamma_2^j \neq 0$;

(5) $h_1 \ominus h_2 \ominus h_3 = h_1 \ominus h_3 \ominus h_2$,

when $\gamma_1 \geqslant \gamma_2, \gamma_1 \geqslant \gamma_3, \gamma_2 \neq 1, \gamma_3 \neq 1$,

and $\gamma_1 - \gamma_2 - \gamma_3 + \gamma_2\gamma_3 \geqslant 0$;

(6) $h_1 \oslash h_2 \oslash h_3 = h_1 \oslash h_3 \oslash h_2$,

when $\gamma_1 \leqslant \gamma_2\gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$;

(7) $h_1 \ominus h_2 \ominus h_3 = h_1 \ominus (h_2 \oplus h_3)$,

when $\gamma_1 \geqslant \gamma_2, \gamma_1 \geqslant \gamma_3, \gamma_2 \neq 1, \gamma_3 \neq 1$, and $\gamma_1-\gamma_2-\gamma_3+\gamma_2\gamma_3\geqslant 0$;

(8) $h_1 \oslash h_2 \oslash h_3 = h_1 \oslash (h_2 \otimes h_3)$,

when $\gamma_1 \leqslant \gamma_2\gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.

Most of the above contents can be extended to interval-valued hesitant fuzzy environments (Chen et al., 2013b).

2.2 Hesitant fuzzy aggregation operators

In this section, we recall some typical aggregation operators.

Definition 4 (Xia and Xu, 2011b). Let $h_i (i=1,2,...,n)$ be a set of HFEs, and $\boldsymbol{\omega} = (\omega_1,\omega_2,...\omega_n)^{\mathrm{T}} (0\leqslant\omega_i\leqslant 1$ and $\sum_{i=1}^{n} \omega_i = 1)$ be the weight vector of them. Then,

(1) The hesitant fuzzy weighted average (HFWA) operator is a mapping from Θ^n to Θ with the following form:

$$HFWA(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}} \omega_i h_i$$

$$= \cup_{\gamma_i \in h_i} \left\{ 1-\prod_{i=1}^{n} (1-\gamma_i)^{\omega_i} \right\}.$$

(2) The hesitant fuzzy weighted geometric (HFWG) operator is a mapping from Θ^n to Θ with the following form:

$$HFWG(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\otimes}} h_i^{\ \omega_i} = \cup_{\gamma_i \in h_i} \left\{ (1-\prod_{i=1}^{n} \gamma_i)^{\omega_i} \right\}.$$

(3) The generalized hesitant fuzzy weighted average (GHFWA) operator is a mapping from Θ^n to Θ with the following form:

$$GHFWA_\lambda(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}} (\omega_i h_i^{\ \lambda})^{1/\lambda}$$

$$= \cup_{\gamma_i \in h_i} \left\{ \left(1-\prod_{i=1}^{n} (1-\gamma_i^\lambda)^{\omega_i} \right)^{1/\lambda} \right\},$$

where $\lambda>0$ is a constant. Particularly, if $\lambda=1$, then the GHFWA operator reduces to the HFWA operator.

(4) The generalized hesitant fuzzy weighted geometric (GHFWG) operator is a mapping from Θ^n to Θ with the following form:

$$GHFWG_\lambda(h_1,h_2,...,h_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{i=1}{\otimes}} (\lambda h_i)^{\omega_i} \right)$$

$$= \cup_{\gamma_i \in h_i} \left\{ 1-\left(1-\prod_{i=1}^{n} \left(1-(1-\gamma_i)^\lambda \right)^{\omega_i} \right)^{1/\lambda} \right\},$$

where $\lambda>0$ is a constant. Particularly, if $\lambda=1$, then the GHFWG operator reduces to the HFWG operator.

Property 3 (Xia and Xu, 2011b). Let $h_i (i=1,2,...,n)$ be a set of HFEs, and $\omega = (\omega_1,\omega_2,...\omega_n)^{\mathrm{T}} (0\leqslant\omega_i\leqslant 1$and $\sum_{i=1}^{n} \omega_i = 1)$ be the weight vector of them. Then, the following three inequalities hold:

(1) $HFWG(h_1,h_2,...,h_n) \leqslant HFWA(h_1,h_2,...,h_n)$;

(2) $HFWG(h_1,h_2,...,h_n) \leqslant GHFWA_\lambda(h_1,h_2,...,h_n)$;

(3) $GHFWG_\lambda(h_1,h_2,...,h_n) \leqslant HFWA(h_1,h_2,...,h_n)$.

Property 4 (Xia and Xu, 2011b). Let $h_i = (i=1,2,...,n)$ be a set of HFEs, and $\boldsymbol{\omega} = (\omega_1,\omega_2,...,\omega_n)^{\mathrm{T}} (0\leqslant\omega_i\leqslant 1)$ and $\Sigma_{i=1}^{n}\omega_i = 1$ be the weight vector of them. Then,

(1) $\oplus_{i=1}^{n} \omega_i h_i^c = (\otimes_{i=1}^{n} h_i^{\omega_i})^c$,

$\otimes_{i=1}^{n} (h_i^c)^{\omega_i} = (\oplus_{i=1}^{n} \omega_i h_i)^c$;

(2) $\left(\oplus_{i=1}^{n} \omega_i (h_i^c)^\lambda\right)^{1/\lambda} = \left(\frac{1}{\lambda} (\otimes_{i=1}^{n} (\lambda h_i)^{\omega_i}) \right)^c$,

$$\frac{1}{\lambda}\left(\overset{n}{\underset{i=1}{\otimes}} (\lambda h_i^c)^{\omega_i} \right) = \left(\left(\overset{n}{\underset{i=1}{\oplus}} (\omega_i h_i^\lambda) \right)^{1/\lambda} \right)^c.$$

Below we introduce some aggregation operators based on ordered aggregation operators:

Definition 5 (Xia and Xu, 2011b). Let $h_i = (i = 1,2,...,n)$ be a set of HFEs, $h_{\sigma(i)}$ be the i^{th} largest one of them, and $\boldsymbol{\omega} = (\omega_1,\omega_2,...,\omega_n)^{\mathrm{T}} (0\leqslant\omega_i\leqslant 1)$ and $\Sigma_{i=1}^{n}\omega_i = 1$ be the vector associated with the integration. Then,

(1) The hesitant fuzzy ordered weighted average (HFOWA) operator is a mapping from to Θ with the following form:

$$HFOWA(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}} \omega_i h_{\sigma(i)}$$

$$= \cup_{\gamma_{\sigma(i)} \in h_{\sigma(i)}} \left\{ 1-\prod_{i=1}^{n} \left(1-\gamma_{\sigma(i)} \right)^{\omega_i} \right\}.$$

(2) The hesitant fuzzy ordered weighted geometric (HFOWG) operator is a mapping from to Θ with the following form:

$$HFOWG(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\otimes}} h_{\sigma(i)}^{\ \ \omega_i}$$

$$= \cup_{\gamma_{\sigma(i)} \in h_{\sigma(i)}} \left\{ 1-\prod_{i=1}^{n} \gamma_{\sigma(i)}^{\ \ \omega_i} \right\},$$

(3) The generalized hesitant fuzzy ordered weighted average (GHFOWA) operator is a mapping from Θ^n to Θ with the following form:

$$GHFOWA_\lambda(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}} \left(\omega_i h_{\sigma(i)}^{\ \ \lambda} \right)^{1/\lambda}$$

$$= \cup_{\gamma_i \in h_{\sigma(i)}} \left\{ \left(1-\prod_{i=1}^{n} \left(1-\gamma_{\sigma(i)}^\lambda \right)^{\omega_i} \right)^{1/\lambda} \right\},$$

where $\lambda>0$ is a constant. Particularly, if $\lambda = 1$, then the GHFOWA operator reduces to the HFOWA operator.

(4) The generalized hesitant fuzzy ordered weighted geometric (GHFOWG) operator is a mapping from Θ^n to Θ with the following form:

$$GHFOWG_\lambda(h_1,h_2,...,h_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{i=1}{\otimes}}\left(\lambda h_{\sigma(i)}\right)^{\omega_i}\right)$$

$$= \cup_{\gamma_{\sigma(i)}\in h_{\sigma(i)}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\gamma_{\sigma(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right\},$$

where $\lambda>0$ is a constant. Particularly, if $\lambda = 1$, then the GHFOWG operator reduces to the HFOWG operator.

Considering both the importance of the HFEs themselves and the importance of the orderly position of the HFEs, we introduce the following four hesitant fuzzy aggregation operators:

Definition 6 (Xia and Xu, 2011b). Let $h_i = (i = 1,2,...,n)$ be a set of HFEs, $\boldsymbol{w} = (w_1,w_2,...w_n)^{\mathrm{T}}(0\leqslant w_i\leqslant 1$ and $\sum_{i=1}^{n} w_i = 1)$ be the weight vector of them, and $\boldsymbol{\omega} = (\omega_1,\omega_2,...\omega_n)^{\mathrm{T}}(0\leqslant\omega_i\leqslant 1$ and $\sum_{i=1}^{n}\omega_i = 1)$ be the vector associated with the integration. Then,

(1) The hesitant fuzzy hybrid average (HFHA) operator is a mapping from to Θ with the following form:

$$HFHA(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}}\omega_i\dot{h}_{\sigma(i)}$$

$$= \cup_{\dot{\gamma}_{\sigma(i)}\in\dot{h}_{\sigma(i)}}\left\{1-\prod_{i=1}^{n}\left(1-\dot{\gamma}_{\sigma(i)}\right)^{\omega_i}\right\},$$

where $\dot{h}_{\sigma(i)}$ is the i^{th} largest one in $\dot{h}_k = nw_kh_k(k = 1,2,...,n)$, and n is the balance factor.

(2) The hesitant fuzzy hybrid geometric (HFHG) operator is a mapping from Θ^n to Θ with the following form:

$$HFHG(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\otimes}}\ddot{h}_{\sigma(i)}{}^{\omega_i}$$

$$= \cup_{\ddot{\gamma}_{\sigma(i)}\in\ddot{h}_{\sigma(i)}}\left\{1-\prod_{i=1}^{n}\ddot{\gamma}_{\sigma(i)}{}^{\omega_i}\right\},$$

where $\ddot{h}_{\sigma(i)}$ is the i^{th} largest one in $\dot{h}_k = h_k^{nw_k}(k = 1,2,...,n)$.

(3) The generalized hesitant fuzzy hybrid average (GHFHA) operator is a mapping from Θ^n to Θ with the following form:

$$GHFHA(h_1,h_2,...,h_n) = \overset{n}{\underset{i=1}{\oplus}}\left(\omega_i\dot{h}_{\sigma(i)}{}^{\lambda}\right)^{1/\lambda}$$

$$= \cup_{\dot{\gamma}_i\in\dot{h}_{\sigma(i)}}\left\{\left(1-\prod_{i=1}^{n}\left(1-\dot{\gamma}_{\sigma(i)}^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right\},$$

where $\lambda>0$ is a constant, $\dot{h}_{\sigma(i)}$ is the i^{th} largest one in. $\dot{h}_k = nw_kh_k$ $(k = 1,2,...,n)$ Particularly, if $\lambda = 1$, then the GHFHA operator reduces to the HFHA operator.

(4) The generalized hesitant fuzzy hybrid geometric (GHFHG) operator is a mapping from to Θ with the following form:

$$GHFHG_\lambda(h_1,h_2,...,h_n) = \frac{1}{\lambda}\left(\overset{n}{\underset{i=1}{\otimes}}\left(\lambda\ddot{h}_{\sigma(i)}\right)^{\omega_i}\right)$$

$$= \cup_{\ddot{\gamma}_{\sigma(i)}\in\ddot{h}_{\sigma(i)}}\left\{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\ddot{\gamma}_{\sigma(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right\},$$

where $\lambda>0$ is a constant, $\ddot{h}_{\sigma(i)}$ is the i^{th} largest one in $\dot{h}_k = h_k^{nw_k}(k = 1,2,...,n)$. Particularly, if $\lambda = 1$, then the GHFHG operator reduces to the HFHG operator.

The above-mentioned are some basic and widely used hesitant fuzzy aggregation operators. In addition to these, the scholars have proposed many other hesitant fuzzy aggregation operators that are more complex and can be applied to more complicated practical decision-making problems. We will not repeat them here for details. As an important part of the research of HFSs, they will be briefly introduced in Section 5.

2.3 Hesitant fuzzy preference relations

Preference relations are an important part of GDM. Sometimes, in a GDM problem, making specific evaluations of all alternatives is difficult. But comparing every two alternatives is relatively simple. At this point, we can use the experts' preferences between the objects (alternatives or attributes) to rank all the objects. This is a general idea that we use the preference relations to solve the GDM problems. As a product of HFSs and preference relations, hesitant fuzzy preference relations (HFPRs) are the representative tool of HFSs applied to GDM. In fact, since the HFPR was introduced, some of its extensions have been proposed. Such as interval-valued hesitant fuzzy preference relation (IVHFPR), probabilistic hesitant fuzzy preference relation (P-HFPR) and probabilistic interval-valued hesitant fuzzy preference relation (P-IVHFPR). They are also widely used in the GDM problems. In this section, some concepts and properties of the HFPR and its extensions will be summarized.

Definition 7 (Xia and Xu, 2013). Let $X = \{x_1,x_2,...,x_n\}$ be a fixed set, then a hesitant fuzzy preference relation can be represented by a matrix $H = (h_{ij})_{n\times n}\in X\times X$, where $h_{ij} = \{\gamma_{ij}^l,l = 1,2,...,|h_{ij}|\}$ is an HFE which expresses all possible preference degrees of the alternative x_i over x_j given by the DMs. Moreover, H should satisfy the conditions below:

$$\gamma_{ij}^{\sigma(l)} + \gamma_{ji}^{\sigma(|h_{ji}|-l+1)} = 1,\ \gamma_{ii} = 0.5,\ |h_{ij}| = |h_{ji}|, i,j = 1,2,...,n,$$

where all $\gamma(i < j = 1,2,...,n)$ are ranked in ascending order and all $\gamma_{ij}(i > j = 1,2,...,n)$ are ranked in descending order. $\gamma_{ij}^{\sigma(l)}$ denotes the l^{th} value in h_{ij}, $l = 1,2,...,|h_{ij}|$, and $|h_{ij}|$ is the number of elements in h_{ij} .

The first condition is reasonable. If the preference degree of the alternative x_i over x_j is γ , then the preference degree of the alternative x_j over x_i should be $1-\gamma$. On the other hand, in the hesitant fuzzy circumstance, the sum of the l^{th} smallest value in h_{ij} and the l^{th} largest value in h_{ji} should be 1. The second condition is easy to understand. The preference degree of the alternative x_i over itself should be 0.5. The third condition means that the length of h_{ij} and h_{ji} should be the same. In particular, if for every h_{ij} ,i , j = 1,2,...,n, there is only one value in it. Then, the HFPR reduces to an FPR.

Example 1. Suppose that lots of experts compare three alternatives in pairs. Experts give the preference degree of the alternative x_1 over x_2. Some experts give 0.2 (then the preference degree of x_2 over x_1 given by them should be $1-0.2 = 0.8$); some of them give 0.25 (then the preference degree of x_2 over x_1 given by them should be $1-0.25 = 0.75$); the others give 0.3 (then the preference degree of x_2 over x_1 given by them should be $1-0.3 = 0.7$). Thus, the preference information of the alternative x_1 over x_2 can be expressed as an HFE $h_{12} = \{0.2,0.25,0.3\}$, and the preference information of the alternative x_2 over x_1 can be expressed as an HFE $h_{21} = \{0.7,0.75,0.8\}$. In a similar way, according to the preference information given by the experts, two HFEs $h_{13} = \{0.4,0.5\}$ and $h_{23} = \{0.55,0.6\}$ can be obtained, then we can get $h_{31} = \{0.5,0.6\}$ and $h_{32} = \{0.4,0.45\}$. Based on the above information, an HFPR $R = (h_{ij})_{n\times n} \in X \times X$ can be obtained as Table 1.

Table 1 The HFPR H

	x_1	x_2	x_3
x_1	{0.5}	{0.2,0.25,0.3}	{0.4,0.5}
x_2	{0.7,0.75,0.8}	{0.5}	{0.55,0.6}
x_3	{0.5,0.6}	{0.4,0.45}	{0.5}

The matrix

$$H = \begin{pmatrix} \{0.5\} & \{0.2,0.25,0.3\} & \{0.4,0.5\} \\ \{0.7,0.75,0.8\} & \{0.5\} & \{0.55,0.6\} \\ \{0.5,0.6\} & \{0.4,0.45\} & \{0.5\} \end{pmatrix},$$

is the preference matrix of the HFPR H.

Similarly, the hesitant multiplicative preference relation (HMPR) can be discussed as follows:

Definition 8 (Xia and Xu, 2013). Let $X = \{x_1,x_2,...,x_n\}$ be a fixed set, then a hesitant multiplicative preference relation can be represented by a matrix $H = (h_{ij})_{n\times n} \in X \times X$, where $h_{ij} = \{\gamma_{ij}^l, l = 1,2,...,|h_{ij}|\}$ is an HFE which expresses all possible preference degrees of the alternative x_i over x_j given by the DMs. Moreover, H should satisfy the conditions below:

$$\gamma_{ij}^{\sigma(l)} \cdot \gamma_{ji}^{\sigma(|h_{ji}|-l+1)} = 1,\ \gamma_{ii} = 1,\ |h_{ij}| = |h_{ji}|, i,j = 1,2,...,n,$$

where all $\gamma_{ij}(i,j = 1,2,...,n)$ are ranked in ascending order, $\gamma_{ij}^{\sigma(l)}$ denotes the l^{th} smallest value in h_{ij}, $l = 1,2,...,|h_{ij}|$ and $|h_{ij}|$ is the number of elements in h_{ij}.

In fact, if the preference degree of the alternative x_i over x_j is γ, then the preference degree of the alternative x_j over x_i should be $1/\gamma$. Thus, in the hesitant multiplicative circumstance, the product of the l^{th} smallest value in h_{ij} and the l^{th} largest value in h_{ji} should be 1. The second condition means that the preference degree of the alternative x_i over itself should be 1. The third condition stipulates that the lengths of h_{ij} and h_{ji} should be the same.

Example 2. Suppose that lots of experts compare three alternatives in pairs. Experts give the preference degree of the alternative x_1 over x_2 with hesitant multiplicative information. Some experts give 1/3 (then the preference degree of x_2 over x_1 given by them should be 1/(1/3)= 3); some of them give 1/2 (then the preference degree of x_2 over x_1 given by them should be 1/(1/2)= 2); the others give 2/3 (then the preference degree of x_2 over x_1 given by them should be 1/(2/3)= 3/2). Thus, the preference information of the alternative x_1 over x_2 can be expressed as an HFE $h_{12} = \{1/3,1/2,2/3\}$, and the preference information of the alternative x_2 over x_1 can be expressed as an HFE $h_{21} = \{3/2,2,3\}$. In a similar way, according to the preference information given by the experts, two HFEs $h_{13} = \{3/4,4,1\}$ and $h_{23} = \{5/4,5/3\}$ can be obtained, then we can get $h_{31} = \{1,4/3\}$ and $h_{32} = \{3/5,4/5\}$. Based on the above information, an HMPR $R = (h_{ij})_{n\times n} \in X \times X$ can be obtained as Table 2.

Table 2 The HMPR H

	x_1	x_2	x_3
x_1	{1}	{1/3,1/2,2/3}	{3/4,1}
x_2	{3/2,2,3}	{1}	{5/4,5/3}
x_3	{1,4/3}	{3/5,4/5}	{1}

The matrix

$$H = \begin{pmatrix} \{1\} & \{1/3,1/2,2/3\} & \{3/4,1\} \\ \{3/2,2,3\} & \{1\} & \{5/4,5/3\} \\ \{1,4/3\} & \{3/5,4/5\} & \{1\} \end{pmatrix}$$ is the preference matrix of the HMPR H.

In many situations, due to certain conditions, the DMs cannot give precise preference degrees to compare the alternatives. Thus, interval-values are good choices to express their evaluations.

Remark. In the above definition, the values in HFEs are required to be arranged in order of sizes. Xu et al. (2017b) pointed out that this condition may cause some problems in the consistency-reaching process.

Definition 9 (Chen et al., 2013b). Let be a fixed set, then an interval-valued hesitant fuzzy preference relation can be expressed by a matrix $\tilde{H} = \left(\tilde{h}_{ij}\right)_{n\times n} \in X \times X$, where $\tilde{h}_{ij} = \left\{\tilde{\gamma}_{ij}^{l}, l = 1,2,...,|\tilde{h}_{ij}|\right\}$ is an IVHFE which expresses all possible preference degrees of the alternative x_i over x_j given by the DMs. Moreover, $\tilde{H}$ should satisfy the conditions below:

$$\inf\tilde{\gamma}_{ij}^{\sigma(l)} + \sup\tilde{\gamma}_{ji}^{\sigma(|h_{ji}|-l+1)} = \sup\tilde{\gamma}_{ij}^{\sigma(l)} + \inf\tilde{\gamma}_{ji}^{\sigma(|h_{ji}|-l+1)} = 1,$$

$$\tilde{\gamma}_{ii} = [0.5,0.5],\ |\tilde{h}_{ij}| = |\tilde{h}_{ji}|, i,j = 1,2,...,n,$$

where all $\tilde{\gamma}_{ij}(i,j = 1,2,...,n)$ are ranked in ascending order, $\tilde{\gamma}_{ij}^{\sigma(l)}$ denotes the l^{th} smallest interval-value in $\tilde{h}_{ij}$, $l = 1,2,...,|\tilde{h}_{ij}|$, and $|\tilde{h}_{ij}|$ is the number of elements in $\tilde{h}_{ij}$.

Similar to the above two definitions, if the preference degree of the alternative x_i over x_j is $\tilde{\gamma} = [\inf\tilde{\gamma}, \sup\tilde{\gamma}]$, then the preference degree of the alternative x_j over x_i should be $[1-\sup\tilde{\gamma}, 1-\inf\tilde{\gamma}]$. Moreover, in the interval-valued hesitant fuzzy circumstance, it should be in line with the elaboration of the first condition. The second condition means that the preference degree of the alternative x_i over itself should be [0.5,0.5]. The third condition stipulates that the length of $\tilde{h}_{ij}$ and $\tilde{h}_{ji}$ should be the same.

Example 3. Suppose that three experts compare the three alternatives in pairs. When they give the preference degree of the alternative x_1 over x_2, one of them gives [0.2,0.3] (then the preference degree of x_2 over x_1 given by him/her should be $[1-0.3, 1-0.2]$= [0.7,0.8]); another one gives [0.3,0.4] (then the preference degree of x_2 over x_1 given by him/her should be $[1-0.4, 1-0.3]$=[0.6,0.7]); the last one gives [0.4,0.5] (then the preference degree of x_2 over x_1 given by him/her should be $[1-0.5, 1-0.4]$ =[0.5,0.6]). Thus, the preference information of the alternative x_1 over x_2 can be expressed as an IVHFE $\tilde{h}_{12} = \{[0.2,0.3],[0.3,0.4],[0.4,0.5]\}$, and the preference information of the alternative x_2 over x_1 can be expressed as an IVHFE $\tilde{h}_{21} = \{[0.5,0.6],[0.6,0.7],[0.7,0.8]\}$. In a similar way, according to the preference information given by the experts, two IVHFEs $\tilde{h}_{13} = \{[0.3,0.4],[0.4,0.5]\}$ and $\tilde{h}_{23} = \{[0.5,0.6],[0.65,0.7]\}$ can be obtained, then we can get $\tilde{h}_{31} = \{[0.5,0.6],[0.6,0.7]\}$ and $\tilde{h}_{32} = \{[0.3,0.35],[0.4,0.5]\}$. Based on the above information, an IVHFPR $\tilde{H} = \left(\tilde{h}_{ij}\right)_{n\times n} \in X \times X$ can be obtained as Table 3.

The matrix

$$\tilde{H} = \begin{pmatrix} \{[0.5,0.5]\} & \{[0.2,0.3][0.3,0.4],[0.4,0.5]\} & \{[0.3,0.4],[0.4,0.5]\} \\ \{[0.5,0.06][0.6,0.7],[0.7,0.8]\} & \{[0.5,0.5]\} & \{[0.5,0.6],[0.65,0.7]\} \\ \{[0.5,0.6],[0.6,0.7]\} & \{[0.3,0.35],[0.4,0.5]\} & \{[0.5,0.5]\} \end{pmatrix}$$

is the preference matrix of the IVHFPR $\tilde{H}$.

Table 3 The HFPR $\tilde{H}$

	x_1	x_2	x_3
x_1	{[0.5,0.5]}	{[0.2,0.3],[0.3,0.4], [0.4,0.5]}	{[0.3,0.4], [0.4,0.5]}
x_2	{[0.5,0.6],[0.6,0.7], [0.7,0.8]}	{[0.5,0.5]}	{[0.5,0.6], [0.65,0.7] }
x_3	{[0.5,0.6],[0.6,0.7] }	{[0.3,0.35],[0.4,0.5] }	{[0.5,0.5]}

3 Consensus building under hesitant fuzzy environment

When dealing with the GDM problems, we need to consider the opinions of all (at least most) experts. If there is a big difference in the opinions of these experts, the decision-making results derived from the information will be not convincing. Thus, the ultimate agreement of the decision-making group's thinking is an important issue that must be taken into account. Therefore, when HFPRs are applied to the GDM problems, the consensus is an important support and guarantee for the GDM process. In this section, we mainly introduce the consensus process of GDM under hesitant fuzzy environment.

3.1 A consensus process of the hesitant fuzzy group decision-making

In this part, we use the method in Ref. Zhang et al. (2015) to explain the general process of a consensus process under hesitant fuzzy environment. At first, the definition of normalized hesitant fuzzy preference relation (NHFPR) is introduced below:

Definition 10 (Zhang et al. 2015a). Suppose that $H = (h_{ij})_{n\times n}$ is an HFPR, $\varsigma(0 \leqslant \varsigma \leqslant 1)$ is the optimized parameter. If we add ς into $h_{ij}(i<j)$, and add $(1-\varsigma)$ into $h_{ji}(i<j)$, then we can obtain a new HFPR $\overline{H} = \left(\overline{h}_{ij}\right)_{n\times n}$ which satisfies the following condition:

(1) $|\overline{h}_{ij}| = \max\left\{|\overline{h}_{ij}|| i,j = 1,2,...,n\right\}, i \neq j$;

(2) $\left(\overline{h}_{ij}\right)^{\sigma(q)} + \left(\overline{h}_{ji}\right)^{\sigma(q)} = 1, \quad h_{ii} = 0.5, i,j = 1,2,...,n$;

(3) $\left(\overline{h}_{ij}\right)^{\sigma(q)} \leqslant \left(\overline{h}_{ij}\right)^{\sigma(q+1)}, \quad \left(\overline{h}_{ji}\right)^{\sigma(q)} \geqslant \left(\overline{h}_{ji}\right)^{\sigma(q+1)}$, where $\left(\overline{h}_{ij}\right)^{\sigma(q)}$ is the q^{th} element of $\overline{h}_{ij}$. Then, $\overline{H} = \left(\overline{h}_{ij}\right)_{n\times n}$ is called the normalized hesitant fuzzy preference relation

based on the optimized parameter s, abbreviated as the NHFPR, $\overline{h}_{ij}$ is called the normalized hesitant fuzzy element (NHFE).

Definition 11 (Zhang et al. 2015a)**.** Let $H = (h_{ij})_{n\times n}$ be an HFPR, and $\overline{H} = (\overline{h}_{ij})_{n\times n}$ be the NHFPR with ς of H. If

$$\overline{h}_{ij} = \overline{h}_{ik} \ominus \overline{h}_{jk} \oplus \left\{\frac{1}{2},\frac{1}{2},...,\frac{1}{2}\right\}, i,j,k = 1,2,...,n,$$

then, H is a consistent HFPR with , where the number of $\frac{1}{2}$ is $|\overline{h}_{ij}|$.

In the above definition, the operation is defined as follows (Zhang et al. (2015a)):

$$h_1 \oplus h_2 = \underset{\gamma_1^{\sigma(s)} \in h_1, \gamma_2^{\sigma(s)} \in h_2}{\cup} \gamma_1^{\sigma(s)} + \gamma_2^{\sigma(s)}$$
$$h_1 \ominus h_2 = \underset{\gamma_1^{\sigma(s)} \in h_1, \gamma_2^{\sigma(s)} \in h_2}{\cup} \gamma_1^{\sigma(s)} - \gamma_2^{\sigma(s)} .$$

Based on the above two definitions, a way to establish a consistent preference relation can be obtained: Let $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$, where $\tilde{h}_{ij} = \frac{1}{n} \oplus_{k=1}^{n} \left(\overline{h}_{ik} \oplus \overline{h}_{kj}\right) \ominus \frac{1}{2}$. Then, $\tilde{H}$ is a consistent HFPR with of H .

Remark. In some conditions, the values obtained according to the above method may be out of scope [0,1], and thus, we may need some other ways to obtain a consistent preference relation (Xu et al., 2018). Due to space limitations, we will not interpret them here.

Zhang et al. (2015a) defined a distance measure based on the Hamming distance between two HFEs as follows:

Definition 12. Let $H_1 = (h_{ij}^1)_{n\times n}$ and $H_2 = (h_{ij}^2)_{n\times n}$ be two HFPRs, $\overline{H}_1 = (\overline{h}_{ij}^1)_{n\times n}$ and $\overline{H}_2 = (\overline{h}_{ij}^2)_{n\times n}$ be their NHFPRs with ς , respectively. Then, the distance between H_1 and H_2 can be defined as follows:

$$D(H_1,H_2) = \frac{2}{n(n-1)} \sum_{i<j}^{n} D_{\mathrm{H}}\left(\overline{h}_{ij}^1, \overline{h}_{ij}^2\right),$$

where $D_{\mathrm{H}}(\overline{h}_{ij}^1, \overline{h}_{ij}^2)$ is the Hamming distance between $\overline{h}_{ij}^1$ and $\overline{h}_{ij}^2$.

Based on Definition 12, a consistency index can be defined below:

Definition 13 (Zhang et al., 2015a). Given an HFPR H, its NHFPR $\overline{H}$ and its consistent HFPR $\tilde{H}$, a consistency index of H can be defined as the distance between $\overline{H}$ and $\tilde{H}$:

$$CI(H) = D(\overline{H}, \tilde{H}).$$

Obviously, the smaller the consistency index $CI(H)$ is, the more consistent the HFPR H is. Especially, $\tilde{H}$ is a consistent HFPR if and only if $CI(H)=0$. Thus, given a threshold $\overline{CI}$, for an HFPR H, if $CI(H) \leqslant \overline{CI}$, then we can call it the HFPR with acceptable consistency. If an HFPR is not an HFPR with acceptable consistency, then we can adjust it in the following way:

For an HFPR $H = (h_{ij})_{n\times n}$ and a threshold $\overline{CI}$, if $CI(H) > \overline{CI}$, then we can construct a new HFPR $H' = (h_{ij}')_{n\times n}$, $h_{ij}' = \delta h_{ij} + (1-\delta)\tilde{h}_{ij}$, where $\delta \in (0,1)$ is a parameter. If $CI(H') \leqslant \overline{CI}$, then the adjustment is completed. Otherwise, repeat the previous step.

After introducing the consistency of the HFPR, we will introduce the consensus index based on the distance between HFPRs. But before this, we need to know the concept of the group hesitant fuzzy preference relation (GHFPR).

Definition 14 (Zhang et al., 2015b)**.** Suppose that $H_k = (h_{ij}^k)_{n\times n} (k = 1,2,...,m)$ are a set of m HFPRs, $\overline{H}_k = \left(\overline{h}_{ij}^k\right)_{n\times n} (k = 1,2,...,m)$ are their NHFPRs, respectively. Let $\boldsymbol{\omega} = (\omega_1,\omega_2,...,\omega_m)^{\mathrm{T}}$ be their weight vector and $\sum_{k=1}^{m} \omega_k = 1$. Then, the group hesitant fuzzy preference relation (GHFPR) is defined as the following form:

$$H_{\mathrm{G}} = \overset{m}{\underset{k=1}{\oplus}} \left(\omega_k \overline{H}_k\right) = \left(\overset{m}{\underset{k=1}{\oplus}} \omega_k h_{ij}^k\right)_{n\times n} .$$

Obviously, according to the definition of GHFPR, it is also an HFPR.

In what follows, we introduce a consensus index to measure the agreement between each individual HFPR and the GHFPR:

Definition 15 (Zhu et al., 2017)**.** Let $H_k = (h_{ij}^k)_{n\times n} (k = 1,2,...,m)$ be a set of HFPRs, s be the optimized parameter, $\overline{H}_k = \left(\overline{h}_{ij}^k\right)_{n\times n} (k = 1,2,...,m)$ be their NHFPRs, respectively, and $H_{\mathrm{G}} = (h_{ij}^{\mathrm{G}})_{n\times n}$ be their GHFPR. Then, the group consensus index of $H_k (k = 1,2,...,m)$ is defined as: $GCI(H_k) = D\left(\overline{H}_k, H_{\mathrm{G}}\right)$.

According to the definition of GHFPR, we can know that the smaller the group consensus index $GCI(H_k)$ is, the more the opinion of the k^{th} expert is consistent with the opinions of the group. If $GCI(H_k) = 0$, then it means that this expert is exactly the same as the group.

Similar to the consistency index, the experts can also give a threshold $\overline{GCI}$. If $GCI(H_k) \leqslant \overline{GCI}$, then H_k and H_{G} are considered to be an acceptable consensus. In practical GDM problems, if $GCI(H_k) \leqslant \overline{GCI}$, then we can adjust H_k with the formula $(h_{ij}^k)' = \delta h_{ij}^k \oplus (1-\delta) h_{ij}^{\mathrm{G}}$ until the acceptable consensus is reached.

In a GDM problem, if m DMs give m preference matrix, then, we can use the above method to reach a consensus among all of them. And then, all the consensus-respected preference information can be used to obtain the ranking of all alternatives. This provides a complete decision support model for a GDM problem.

3.2 Supplementary explanation

The consensus process under hesitant fuzzy environment is mainly divided into two parts: the consensus process and the selection process. Based on the characteristics of the consensus process, the research on it mainly focuses on the consensus process, especially the consistency and consensus indices and the feedback mechanisms.

Xu et al. (2017b) proposed a consensus model for HFPRs. Its core content is to establish a new hesitant fuzzy consistency measure and a consensus measure with a new standardized method of HFEs proposed by themselves. They also gave two new feedback mechanisms. Zhang et al. (2018b) improved the additive consistency by using some mixed 0–1 linear programming models. Some new consistency models were established to supplement the missing elements for incomplete HFPRs. He and Xu (2017) proposed a consensus reaching model with different hesitant preference structures. A distance measure based on three kinds of hesitant preference structures was introduced, and an interaction mechanism was proposed to help the DMs adjust their evaluation values. Wu and Xu (2018) proposed a consensus model for large-scale GDM with hesitant fuzzy information and changeable clusters. In their researches, a distance measure was given to compute the various consensus measures, and a new feedback mechanism based on the changeable clusters was proposed.

In addition, Liao et al. (2014) proposed a complete consensus process of the HMPR. At first, the definition of the consistency of the HMPR and the complete multiplicative consistent HFPR were proposed. Using the above concepts, a modified algorithm for the consistency of HMPRs was obtained. Moreover, they used the Hamming distance to give the consensus measure between different HMPRs in the GDM problems, and gave a feedback mechanism to correct the inconsistent HFPRs. The above is the whole process of the hesitant multiplicative GDM consensus process. It is a very important supplement to the consensus process under hesitant fuzzy environment.

In a word, there is not much research on the consensus process under hesitant fuzzy environment, and there is still a lot of room for development in this area.

3.3 Hesitant analytic hierarchy process

In 2016, Zhu et al. (2016) proposed the hesitant analytic hierarchy process (H-AHP) which can consider the hesitancy of the DMs to enhance the modeling ability of traditional AHP based on the consistency and consensus of HFPRs and HMPRs. The general process of the H-AHP is as follows:

(1) For a control criterion, it needs to establish a control level, a criterion level, and an alternative level to configure a top-to-bottom hierarchical structure, where each of these levels may have sublevels;

(2) For each objective in the upper level, let the DMs compare the objectives in low-levels that correspond to them to establish HCMs from the information given by the DMs;

(3) Establish the consistency index, and then check the consistency degree of each individual HCM to confirm whether they are acceptable. After that, it needs to adjust them through a certain feedback mechanism until they all become acceptable;

(4) Establish the consensus index, and then check the consensus degrees of the HCMs in each level to confirm whether they are acceptable. After that, it needs to improve them through a certain feedback mechanism until they are acceptable;

(5) For the objectives in each level of the hierarchical structure, we derive their holistic priorities;

(6) Integrate the holistic priorities of all alternatives in the order from bottom level to top level. Finally, the holistic priorities of all alternatives with respect to the control criterion can be obtained.

The content of hesitant fuzzy analytic hierarchy is very rich. Due to space limitations, this section is only an overview of its process. More detailed content, such as the establishment of consensus index and feedback mechanisms, can be seen in Zhu and Xu (2014) and Zhu et al. (2016).

4 Group decision-making methods based on HFSs

4.1 Hesitant fuzzy TOPSIS

In order to better solve the multi-attribute group decision-making (MAGDM) problems, Xu and Zhang (2013) extended the TOPSIS method to the hesitant fuzzy environment. Its main idea is to use the distance between each alternative and the hesitant fuzzy positive and negative ideal solutions (HF-PIS and HF-NIS) to rank the alternatives. The HF-PIS and HF-NIS can be defined as follows:

$$
\begin{aligned}
A^{+} &= \left\{ x_j, \max_i \langle \gamma_{ij}^{\sigma(\lambda)} \rangle | j = 1,2,\dots,n \right\} \\
&= \{ \langle x_1, ((\gamma_1^1)^+, (\gamma_1^2)^+, \dots, (\gamma_1^l)^+) \rangle, \\
&\quad \langle x_2, ((\gamma_2^1)^+, (\gamma_2^2)^+, \dots, (\gamma_2^l)^+) \rangle, \dots, \\
&\quad \langle x_n, ((\gamma_n^1)^+, (\gamma_n^2)^+, \dots, (\gamma_n^l)^+) \rangle \},
\end{aligned}
$$

$$A^- = \left\{x_j, \min_i \langle \gamma_{ij}^{\sigma(\lambda)} \rangle | j = 1,2,...,n\right\}$$
$$= \{\langle x_1,((\gamma_1^1)^-,(\gamma_1^2)^-,...,(\gamma_1^l)^-)\rangle,$$
$$\langle x_2,((\gamma_2^1)^-,(\gamma_2^2)^-,...,(\gamma_2^l)^-)\rangle,...,$$
$$\langle x_n,((\gamma_n^1)^-,(\gamma_n^2)^-,...,(\gamma_n^l)^-)\rangle\}.$$

After obtaining the HF-PIS and the HF-NIS, we can use the hesitant fuzzy Euclidean distances (Xu and Xia, 2011) to calculate the distances between the alternatives and them. Then we can get:

$$d_i^+ = \sum_{j=1}^n d(h_{ij},h_j^+)w_j = \sqrt{\frac{1}{l}\sum\nolimits_{\lambda=1}^l |h_{ij}^{\sigma(\lambda)} - (h_j^{\sigma(\lambda)})^+|^2},$$
$$i = 1,2,...,n;$$
$$d_i^- = \sum_{j=1}^n d(h_{ij},h_j^-)w_j = \sqrt{\frac{1}{l}\sum\nolimits_{\lambda=1}^l |h_{ij}^{\sigma(\lambda)} - (h_j^{\sigma(\lambda)})^-|^2},$$
$$i = 1,2,...,n.$$

The relative closeness coefficient of an alternative X_i and the HF-PIS is defined as follows:

$$C_i = \frac{d_i^-}{d_i^+ + d_i^-}.$$

Obviously, the higher the relative closeness coefficient C_i, the closer the distance is to the HF-PIS. Then, we can use the relative closeness coefficients to rank all alternatives. The above is the core idea of the hesitant fuzzy TOPSIS method.

Xu and Zhang (2013) also extended this method to the interval-valued hesitant fuzzy environment, which makes this method further improved. Sun et al. (2018) proposed an improved synthetic correlation coefficient and applied it to hesitant fuzzy environment to form an innovative hesitant fuzzy TOPSIS approach.

4.2 Hesitant fuzzy PROMETHEE

PROMETHEE is an effective method to solve the MAGDM problems. The main idea of PROMETHEE is to use the deviation degree of each pair of programs on a particular attribute to establish preference degrees between different alternatives. Through these preference degrees, the overall preference degree between each two alternatives is obtained, and the outgoing flow, the entering flow and the net flow of each alternative are further obtained. Therefore, we can rank all alternatives with them.

Mahmoudi et al. (2016) applied hesitant fuzzy elements to PROMETHEE and established hesitant fuzzy PROMETHEE. It allows the DMs to apply HFEs when giving evaluation values for each alternative with respect to different attributes. Then, we can obtain preference degrees between every two alternatives with HFEs and establish an HFPR. It avoids the deviation of decision-making results caused by the DMs forcibly giving accurate evaluation values under uncertain conditions.

Hesitant fuzzy PROMETHEE can be roughly summarized as the following steps:

(1) For a given decision-making problem, there are a set of alternatives $X = \{X_1,X_2,...,X_m\}$ and a set of attributes $A = \{A_1,A_2,...,A_n\}$;

(2) The DMs give evaluation values of the alternatives in $X = \{X_1,X_2,...,X_m\}$ with respect to the attributes in $C = \{A_1,A_2,...,A_n\}$ respectively with HFEs. At the same time, it needs to determine the weight vector of all attributes, $\boldsymbol{\omega} = (\omega_1,\omega_2,...,\omega_n)^{\mathrm{T}}$, where $\sum\nolimits_{j=1}^n \omega_j = 1$;

(3) Compute the deviation degrees of every pair of alternatives with respect to different attributes $A = \{A_1,A_2,\cdots,A_n\}$. We determine the DM's preference function, that is, determine the no-difference-threshold q and the strict-preference-threshold p, and then, compute the preference degree μ_{ij}^k between the alternatives X_i and X_j with respect to the attribute A_k through the linear preference function (V-shape), and get the preference matrix $\boldsymbol{U}^k(k = 1,2,...,n)$;

(4) Establish the HFPR $H^k = (\mu_{ij}^k)_{m\times m}$ with respect to the attributes $\boldsymbol{A}_k(k = 1,2,...,n)$;

(5) Establish the overall HFPR $\boldsymbol{H} = (r_{ij})_{m\times m}$;

(6) Compute the outgoing flows and entering flows of all alternatives $X_i(i = 1,2,...,m)$, denoted as $\varphi^+(X_i)$ and $\varphi^-(X_i)$;

(7) Compare the sizes of all $\varphi^+(X_i)(i = 1,2,...,m)$ and $\varphi^-(X_i)(i = 1,2,...,m)$ to get the preference orders of all alternatives.

The above is just a rough explanation of hesitant fuzzy PROMETHEE, please see the detailed algorithm and formulas in Mahmoudi et al. (2016).

4.3 Hesitant fuzzy ELECTRE

In traditional ELECTRE I methods, for different alternatives X_i and X_j , we can divide the attribute set into two different subsets: the concordance set and the discordance set. The concordance set contains the attributes for which X_i is preferred X_j ; the discordance set contains those attributes for which X_i is inferior to X_j . However, in hesitant fuzzy circumstance, benefitted from the score value and the deviation degree, the hesitant fuzzy concordance (discordance) set can be divided into the hesitant fuzzy concordance (discordance) set and the weakened hesitant fuzzy concordance (discordance) set.

Chen et al. (2015) explained these concepts in detail. Let $A = \{A_1,A_2,\cdots,A_n\}$ be an attribute set. $X = \{X_1,X_2,\cdots,X_m\}$ be the alternative set, and h_{ik} be an HFE, which expresses the possible evaluation values of the alternative X_i under the attribute A_k. For each pair of alternatives $X_i, X_j (i,j = 1,2,\cdots,m; i \neq j)$, their hesitant fuzzy concordance set contains the attributes for which X_i is preferred to X_j, and it can be represented as follows:

$$J_{c_{ij}} = \{k | s(h_{ik}) \geqslant s(h_{jk}) \text{ and } \overline{\sigma}(h_{ik}) < \overline{\sigma}(h_{jk})\},$$

and the weakened hesitant fuzzy concordance set of them is defined as:

$$J_{c'_{ij}} = \{k | s(h_{ik}) \geqslant s(h_{jk}) \text{ and } \overline{\sigma}(h_{ik}) \geqslant \overline{\sigma}(h_{jk})\},$$

where $s(h_{ik})$ and $\overline{\sigma}(h_{ik})$ are the score value and the deviation degree of h_{ik}, respectively. The computing methods of them can be seen in Xia and Xu (2011).

The difference of $J_{c_{ij}}$ and $J_{c'_{ij}}$ is the deviation degree. The lower deviation degree expresses that the opinions of different DMs have higher consistency. Therefore, $J_{c_{ij}}$ has higher consistency than $J_{c'_{ij}}$.

Similarly, the hesitant fuzzy discordance set of X_i and X_j contains the attributes for which X_i is inferior to X_j, and it can be represented as follows:

$$J_{d_{ij}} = \{k | s(h_{ik}) < s(h_{jk}) \text{ and } \overline{\sigma}(h_{ik}) \geqslant \overline{\sigma}(h_{jk})\},$$

and the weakened hesitant fuzzy discordance set of them is defined as:

$$J_{d'_{ij}} = \{k | s(h_{ik}) < s(h_{jk}) \text{ and } \overline{\sigma}(h_{ik}) < \overline{\sigma}(h_{jk})\}.$$

Obviously, $J_{d_{ij}}$ has higher consistency than $J_{d'_{ij}}$.

The above is the core content of HF-ELECTRE I, we will outline this method below (Readers can find the detailed algorithm and formulas in Chen et al. (2015)):

(1) Establish the hesitant fuzzy decision-making matrix. We determine the weight vector of all attributes $\boldsymbol{\omega} = (\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ and the attitude weight vector of different kinds of hesitant fuzzy concordance (discordance) set $\boldsymbol{\omega} = (\omega_c,\omega_{c'},\omega_{\overline{D}},\omega_{\overline{D}}{}')^{\mathrm{T}}$;

(2) Compute the score values and the deviation degrees of the alternatives under all attributes;

(3) Construct the hesitant fuzzy concordance (discordance) set and the weakened hesitant fuzzy concordance (discordance) set;

(4) Compute the hesitant fuzzy concordance (discordance) index and the weakened hesitant fuzzy concordance (discordance) index. We establish the hesitant fuzzy concordance (discordance) matrix and the weakened hesitant fuzzy concordance (discordance) matrix;

(5) Determine the consistent dominant matrix and the inconsistent dominant matrix;

(6) Establish the dominant aggregation matrix;

(7) Draw the decision-making map and choose the better alternative.

After the hesitant fuzzy ELECTRE I was introduced, Chen and Xu (2015) proposed the hesitant fuzzy ELECTRE II. Recently, Galo et al. (2018) proposed the hesitant fuzzy ELECTRE TRI and applied it to a supplier categorization problem.

4.4 Hesitant fuzzy VIKOR

The VIKOR is a method for solving the MAGDM problems with conflicting properties based on the special measure of the closeness degree of the alternative to the ideal solution. In many decision-making situations, especially in the beginning of the decision-making process, the DMs may prefer to use a list of values to express their evaluation information instead of using isolated deterministic values. Moreover, when the DMs give their evaluation values for some alternatives with respect to different attributes, they often have difficulty in reaching agreement. Therefore, it is more reasonable to use HFEs to express the evaluation information given by the DMs. Based on the above analysis, Liao and Xu (2013) proposed the hesitant fuzzy VIKOR (HF-VIKOR).

Remark. Attributes are often divided into the benefit-type attributes and the cost-type attributes. Here we only discuss the benefit-type attributes, and the relevant content of the cost-type attributes is similar.

The first step of the HF-VIKOR is to find the ideal solution. If we compute the score values and the deviation degrees of all HFEs in the hesitant fuzzy decision-making matrix and compare them, we can obtain the optimal value and the worst value of the alternative X_i with respect to the attribute A_j, denoted as:

$$\begin{cases} h_j^* = \max\limits_i h_{ij} \\ h_j^- = \min\limits_i h_{ij} \end{cases}.$$

The next step is to find the hesitant fuzzy group utility measure and the individual regret measure. Before it, the Manhattan L_p-metric of HFEs should be defined.

Definition 16 (Liao and Xu, 2013)**.** The hesitant fuzzy Manhattan L_p-metric is defined as:

$$L_{p,i} = \left(\sum_{j=1}^{n}\left(\omega_j \frac{d(h_j^*,h_{ij})}{d(h_j^*,h_j^-)}\right)^p\right)^{1/p}$$

$$(1 \leqslant p \leqslant +\infty; i = 1,2,\ldots,m),$$

where $\omega_j (j = 1,2,\ldots,n)$ is the weight of the attribute A_j, which satisfies $0 \leqslant \omega_j \leqslant 1 (j = 1,2,\ldots,n)$ and $\sum_{j=1}^{n} \omega_j = 1$; $d(h_j^*,h_{ij})$ is the Manhattan distance between h_j^* and h_{ij}, which satisfies:

$$d(h_j^*,h_{ij}) = \frac{1}{l_j}\sum_{t=1}^{l_j} |h_j^{*\,\sigma(t)} - h_{ij}^{\ \sigma(t)}|,$$

where $h_j^{*\sigma(t)}$ and $h_{ij}^{\sigma(t)}$ are the t^{th} largest elements of h_j^* and h_{ij}, respectively, and $l_j = \max\{|h_j^*|,|h_{ij}|\}.d(h_j^*,h_j^-)$ can be defined in a similar way.

According to Definition 5, the hesitant fuzzy group utility measure can be expressed as the following form:

$$S_i = L_{1,i} = \sum_{j=1}^{n}\left(\omega_j \frac{d(h_j^*,h_{ij})}{d(h_j^*,h_j^-)}\right).$$

Similarly, the hesitant fuzzy individual regret measure can be defined as:

$$R_i = L_{+\infty,i} = \max_j\left(\omega_j \frac{d(h_j^*,h_{ij})}{d(h_j^*,h_j^-)}\right).$$

Based on the above content, the hesitant fuzzy compromise measure can be obtained:

Definition 17 (Liao and Xu, 2013)**.** The hesitant fuzzy compromise measure is defined as the form:

$$Q_i = v\frac{S_i - S^*}{S^- - S^*} + (1-v)\frac{R_i - R^*}{R^- - R^*},$$

where $S^* = \min_i S_i$, $S^- = \max_i S_i$, $R^* = \min_i R_i$, $R^- = \max_i R_i$; v is the weight of the strategy of maximizing the utility of attributes. The greater its value, the more average the DM's preferences on different attributes will be. v is usually taken as 0.5 without loss of generality.

It can be seen that the hesitant fuzzy compromise measure consists of two parts. The core of the first half is hesitant fuzzy group utility measure, and the core of the second half is hesitant fuzzy individual regret measure. The smaller the value of the hesitant fuzzy compromise measure, the better the alternative. Thus, the ranking of all alternatives can be obtained.

4.5 Hesitant fuzzy TODIM

The widely used MAGDM methods, such as TOPSIS and ELECTRE, usually assume that the DMs have complete rationality. However, in the actual decision-making processes, the DMs are usually not completely rational (Camerer, 1998). Thus, the psychological behaviors of the DMs will have an important impact on the decision-making processes. Therefore, considering the psychological behavioral factors of the DMs, Zhang and Xu (2014b) proposed the hesitant fuzzy TODIM approach based on a new measure function and applied it to the MAGDM problems. Below we first introduce this new measure function of HFEs:

Definition 18 (Zhang and Xu, 2014b)**.** Let $h = \{\gamma_1,\gamma_2,...\gamma_n\}$ be an HFE, a new measure function $Z_\delta(h)$ of h can be defined as follows:

$$Z_\delta(h) = Z_\delta(\gamma_1,\gamma_2,...,\gamma_n) = \left(\frac{(\gamma_1)^\delta + (\gamma_2)^\delta + ... + (\gamma_n)^\delta}{n}\right)^{1/\delta},$$

where δ is a parameter given by the DMs, which can be changed according to the specific decision-making problems, and satisfies that $0 < \delta \leqslant 1$. In particular, if $\delta = 1$ then $Z_\delta(h)$ reduces to the score function (Xia and Xu, 2011) of HFEs.

With the above definition, a new way to rank the HFEs can be obtained as follows: For two HFEs h_1 and h_2,

(1) If $Z_\delta(h_1) > Z_\delta(h_2)$, then h_1 is superior to h_2, denoted as $h_1 > h_2$;

(2) If $Z_\delta(h_1) < Z_\delta(h_2)$, then h_1 is inferior to h_2, denoted as $h_1 < h_2$;

(3) If $Z_\delta(h_1) = Z_\delta(h_2)$, then h_1 is equivalent to h_2, denoted as $h_1 \sim h_2$.

Based on the above measure function and the HFE-ranking method, in the following, we introduce the hesitant fuzzy TODIM:

First of all, similar to the typical TODIM method, the attribute with the highest weight A_{i_0} is considered as the reference attribute. Then, the relative weight of each attribute A_i relative to the reference attribute A_{i_0} can be computed with:

$$w_{ii_0} = \frac{w_i}{w_{i_0}},\ i = 1,2,...,n.$$

After getting the above information, the gain and loss values of the alternative X_i relative to X_j with respect to different attributes can be computed. Let $A = \{A_1,A_2,..., A_n\}$ be an attribute set, and $X = \{X_1,X_2,...,X_m\}$ be the alternative set, the evaluation value of the alternative X_i with respect to the attribute A_k be h_{ik}, $\boldsymbol{w} = (w_1,w_2,...,w_n)^{\mathrm{T}}$ be the weight vector of A, and w_{k_0} be the largest one among $w_k (k = 1,2,...,n)$. Then, the perceptual function value of X_i relative to X_j on A_k is:

$$\varphi_k(X_i,X_j) = \begin{cases} \sqrt{w_{kk_0} d(h_{ik},h_{jk}) / \sum_{k=1}^{n} w_{kk_0}}, & \text{if } Z_\delta(h_{ik}) - Z_\delta(h_{jk}) > 0, \\ 0, \quad \text{if } Z_\delta(h_{ik}) - Z_\delta(h_{jk}) = 0, & \\ -\frac{1}{\theta}\sqrt{\left(\sum_{k=1}^{n} w_{kk_0}\right) d(h_{ik},h_{jk}) / w_{kk_0}}, & \text{if } Z_\delta(h_{ik}) - Z_\delta(h_{jk}) < 0, \end{cases}$$

where θ expresses the attenuation factor of the losses, $d(h_{ik},h_{jk})$ is the Euclidean distance measure of h_{ik} and h_{jk}. If $Z_\delta(h_{ik})-Z_\delta(h_{jk})>0$, then $\varphi_k(X_i,X_j)$ represents "gains"; If $Z_\delta(h_{ik})-Z_\delta(h_{jk})=0$, then $\varphi_k(X_i,X_j)$ represents a "nil"; If $Z_\delta(h_{ik})-Z_\delta(h_{jk})>0$, then $\varphi_k(X_i,X_j)$ represents "losses".

Next, we can integrate the perceptual function of X_i relative to X_j under all attributes in $A=\{A_1,A_2,...,A_n\}$:

$$\vartheta(X_i,X_j)=\sum_{k=1}^{n}\varphi_k(X_i,X_j),\ i,j=1,2,...,m.$$

Finally, the overall perceptual function values of $X_i(i=1,2,...,m)$ can be computed with the following formula:

$$\Phi(X_i)=\frac{\sum_{j=1}^{m}\vartheta(X_i,X_j)-\min_i\{\sum_{j=1}^{m}\vartheta(X_i,X_j)\}}{\max_i\{\sum_{j=1}^{m}\vartheta(X_i,X_j)\}-\min_i\{\sum_{j=1}^{m}\vartheta(X_i,X_j)\}},$$

$$i=1,2,...,m.$$

It is obvious that $0\leqslant\Phi(X_i)\leqslant1$. The greater the overall perceptual function value of the alternative X_i, the better the alternative X_i. Thus, the alternatives X_i $(i=1,2,...,m)$ can be ranked with the ranking of their overall perceptual function values.

In the same reference, the authors also proposed the interval-valued hesitant fuzzy TODIM method. It allows the DMs to have greater hesitant degree in giving initial decision-making information. These two methods have become an important part of the classical decision-making approach — TODIM.

4.6 Hesitant fuzzy QUALIFLEX

For a non-single value HFE, the membership degrees in it always have a deviation. It can represent the hesitant degree of the DM giving this evaluation value. In order to express the deviation, the hesitant index of HFEs can be defined below:

Definition 19 (Zhang and Xu, 2015)**.** For a given HFE $h=\{\gamma_i|i=1,2,...,n\}$, its hesitant index is defined as:

$$\overline{h}(h)=\sum_{i,j=1}^{n}\left(\gamma_{\sigma(j)}-\gamma_{\sigma(i)}\right)/C_n^2,$$

where $\gamma_{\sigma(i)}$ and $\gamma_{\sigma(j)}$ are the i^{th} and j^{th} largest membership degrees in h and $j>i$.

It is easy to see that if there is only one membership degree in h, then $\overline{h}(h)=0$. This means that the DM does not hesitate to give this evaluation value.

Based on the hesitant index, we can obtain a new way to compare the sizes of different HFEs:

$$h_1\leqslant h_2\Leftrightarrow\gamma_1^{\sigma(l)}\leqslant\gamma_2^{\sigma(l)}(i=1,2,...,n)\text{ and }\overline{h}(h_1)\geqslant\overline{h}(h_2).$$

According to the above method, it can be seen that $h=\{1\}$ is the largest HFE, therefore, $h=\{1\}$ is called the ideal HFE, denoted as $\tilde{1}$.

In the process of dealing with hesitant fuzzy information, hesitant index plays an important role. Considering the influence of hesitant index, some new ranking methods and distance measures of HFEs will be proposed below:

Definition 20 (Zhang and Xu, 2015). For two given HFEs $h_i\{\gamma_i^{\sigma(l)}|l=1,2,...,|h_i|\}(i=1,2)$, let $|h_1|=|h_2|=n$, then the improved hesitant fuzzy Hamming distance can be defined as:

$$d_{\text{NH}}(h_1,h_2)=\frac{1}{2}\left(\frac{1}{n}\sum_{l=1}^{n}|\gamma_1^{\sigma(l)}-\gamma_2^{\sigma(l)}|+|\overline{h}(h_1)-\overline{h}(h_2)|\right),$$

and the improved hesitant fuzzy Euclidean distance can be defined below:

$$d_{\text{NE}}(h_1,h_2)=\sqrt{\frac{1}{2}\left(\frac{1}{n}\sum_{l=1}^{n}\left(\gamma_1^{\sigma(l)}-\gamma_2^{\sigma(l)}\right)^2+\left(\overline{h}(h_1)-\overline{h}(h_2)\right)^2\right)}.$$

Based on the improved Hamming distance, the signed distance of a given HFE $\{\gamma_i|i=1,2,...,n\}$, i.e., the Hamming distance between it and the ideal HFE $\tilde{1}$ can be obtained:

$$d_{\text{S}}(h,\tilde{1})=\frac{1}{2}\left(\frac{1}{n}\sum_{l=1}^{n}(1-\gamma^l)+\overline{h}(h)\right).$$

Moreover, we can get a way to rank HFEs based on the signed distance. For two HFEs h_1 and h_2,

(1) If $d_{\text{S}}(h_1,\tilde{1})<d_{\text{S}}(h_2,\tilde{1})$, then h_1 is inferior to h_2, denoted as $h_1\prec h_2$;

(2) If $d_{\text{S}}(h_1,\tilde{1})>d_{\text{S}}(h_2,\tilde{1})$, then h_1 is superior to h_2, denoted as $h_1\succ h_2$;

(3) If $d_{\text{S}}(h_1,\tilde{1})=d_{\text{S}}(h_2,\tilde{1})$, then h_1 is equivalent to h_2, denoted as $h_1\sim h_2$.

In the traditional QUALIFLEX method, we should obtain all the results of the orders of all alternatives at first, and then test them one by one. In the test process, we need to compare the size of the evaluation values for each of the two alternatives under each attribute to determine if their orders are consistent with the ranking in the test orders. Then we can compute the consistency (non-consistency) index for the ranking of each two alternatives.

However, in the hesitant fuzzy circumstance, the evaluation values of each alternative with respect to all attributes are HFEs. Sometimes, even the weights of the attributes are HFEs. It is hard to compare them directly. Therefore, we use the ranking method of HFEs based on the signed distance to determine the consistency (non-consistency) index for the ranking of each two alternatives. Below we consider a problem that the weights of attributes are also HFEs:

Suppose that the l^{th} descending order of the alternatives is: $P_l=(...,X_i,...X_j,...)$. Then, the consistency index of the ranking (X_i,X_j) with respect to the attribute A_k can be defined as:

$$\varphi_k^l(X_i,X_j) = d_S(h_{jk},\tilde{1}) - d_S(h_{ik},\tilde{1}).$$

Then, according to the ranking method of HFEs based on the signed distance, the following conclusions can be obtained:

If $\varphi_k^l(X_i,X_j) \geqslant 0$, i.e., $d_S\left(h_{ik},\tilde{1}\right) \leqslant d_S\left(h_{jk},\tilde{1}\right)$, then we can think that X_i is in front of X_j under the attribute A_k . That is, it is inconsistent with the order of the two in the ranking we tested. Therefore, in this condition, $\varphi_k^l(X_i,X_j)$ is called the non-consistency index of the order (X_i,X_j) under the attribute A_k .

If $\varphi_k^l(X_i,X_j) < 0$, i.e., $d_S\left(h_{ik},\tilde{1}\right) > d_S\left(h_{jk},\tilde{1}\right)$, then we can think that X_i is behind of X_j under the attribute A_k. That is, it is consistent with the order of the two in the ranking we tested. Therefore, in this condition, $\varphi_k^l(X_i,X_j)$ is called the consistency index of the order (X_i,X_j) under the attribute A_k .

Let the weight vector of the attributes be $\tilde{W} = \left(\tilde{W}_1,\tilde{W}_2,...\tilde{W}_n\right)^{\mathrm{T}}$, where $\tilde{W}_i(i=1,2,...,n)$ are all HFEs. We can compute the weighted consistency (non-consistency) index of the order (X_i,X_j) under the l^{th} order $P_l = (...,X_i,...X_j,...)$:

$$\varphi^l(X_i,X_j) = \sum_{k=1}^{n} \varphi_k^l\left(X_i,X_j\right)\left(1 - d_S\left(\tilde{w}_k,\tilde{1}\right)\right).$$

Finally, the overall consistency (non-consistency) index of the l^{th} order $P_l = (...,X_i,...X_j,...)$ can be defined as follows:

$$\varphi^l = \sum_{X_i,X_j \in X} \sum_{k=1}^{n} \varphi_k^l\left(X_i,X_j\right)\left(1 - d_S\left(\tilde{w}_k,\tilde{1}\right)\right).$$

It is easy to see that the larger the overall consistency (non-consistency) index φ^l , the more reasonable the l^{th} order $P_l = (...,X_i,...X_j,...)$. Therefore, the best ranking of the alternatives can be determined by comparing the sizes of the overall consistency (non-consistency) indexes with

$$P^* = \max_{l=1}^{m!}\{\phi^l\}.$$

4.7 Hesitant fuzzy LINMAP

The MAGDM process has several forms. One of the common forms is that multiple experts give their evaluation values over each alternative with respect to all attributes, and we rank all alternatives by using some methods with those evaluation values. When the evaluation values given by the experts are HFEs, the problem will become more complicated.

Zhang and Xu (2014a) proposed the hesitant fuzzy LINMAP method based on the interval programming model. It is mainly used to solve the following MAGDM problems:

(1) The evaluation values given by the DMs are HFEs;

(2) The preferences between the alternatives given by the DMs are interval-values;

(3) The weights of attributes are partially known or completely unknown.

Considering a MAGDM problem: Suppose that $X = \{X_1,X_2,...,X_m\}$ is the alternative set, $A = \{A_1,A_2,...,A_n\}$ is the attribute set, $E = \{E_1,E_2,...,E_l\}$ is the expert set. The weight vector of the attributes is $\boldsymbol{w} = (w_1,w_2,...,w_n)^{\mathrm{T}}$ and not all $w_1(i=1,2,...n)$ are known. The weights of all experts are the same. The evaluation value of the alternative X_i with respect to the attribute A_j given by the expert E_k is represented as $h_j^k(x_i)$. The comparing preference degrees of all pairs of the alternatives $(X_i,X_j) \times (i,j=1,2,...m)$ which are given by the expert E_k are denoted as the interval-values $\tilde{C}_k(i,j)(i,j=1,2,...m)$.

In order to deal with the above problem, the interval consistency (non-consistency) index needs to be defined.

First of all, suppose that the hesitant fuzzy positive ideal solution is $X^* = (h_1^*,h_2^*,...h_n^*)$, where $h_i^* = \{(\gamma_i^1)^*,(\gamma_i^2)^*,...(\gamma_i^{|h_i^*|})^*\}(i=1,2,...,n)$ are HFEs. For the expert E_k, the square of the weighted Euclidean distance between $X_i \in X$ and X^* are denoted as D_i^k, $I = 1,2,...,m$, i.e.,

$$D_i^k = \sum_{j=1}^{n} w_j d_E(h_{ij}^k,h_j^*)^2 = \sum_{j=1}^{n} w_j \left(\frac{1}{|h_j^*|}\sum_{l=1}^{|h_j^*|}\left((\gamma_{ij}^l)^k - (\gamma_j^l)^*\right)^2\right),$$

$$i = 1,2,...,m.$$

Thus, for two alternatives X_i and X_j , the weighted Euclidean distance between X^* and them can be computed, denoted as D_i^k and . Let $\rho_{ij}^k = D_i^k - D_j^k$, then, for the pair of alternatives (X_i,X_j) with the order $X_i \succ X_j$ which is given by the experts, we have:

If $\rho_{ij}^k \leqslant 0$, i.e., $D_i^k \leqslant D_j^k$, then X_i is closer than X_j from X^*. It means that X_i is superior to X_j . That is, this is consistent with the order given by the expert.

If $\rho_{ij}^k \geqslant 0$, i.e., $D_i^k \geqslant D_j^k$, then X_i is farther than X_j from X^*. It means that X_i is inferior to X_j . That is, this is inconsistent with the order given by the experts.

In order to measure the consistency degree between the order of (X_i,X_j) determined by the distance (D_i^k,D_j^k) and the order given by the expert E_k, the consistency index should be determined at first:

$$(D_i^k - D_j^k)^* = \begin{cases} \tilde{C}_k\left(i,j\right)\cdot(D_j^k - D_i^k), & \text{if } D_i^k \leqslant D_j^k, \\ 0, & \text{if } D_i^k > D_j^k. \end{cases}$$

Let Ω_k be the set of all alternative pairs. Then, the consistency index of all alternative pairs under the expert E_k can be defined as follows:

$$\tilde{G}_k{}^* = \sum_{(i,j)\in\Omega} (D_i^k - D_j^k)^*.$$

Therefore, the group consistency index under all the experts can be expressed as:

$$\tilde{G}^* = \sum_{k=1}^{l} \tilde{G}_k^*.$$

Similarly, the inconsistency index can be determined below:

$$(D_i^k - D_j^k)^- = \begin{cases} 0, & \text{if } D_i^k \geqslant D_j^k, \\ \tilde{C}_k\left(i,j\right)\cdot(D_j^k - D_i^k), & \text{if } D_i^k < D_j^k. \end{cases}$$

We can also get the group inconsistency index under all the experts:

$$\tilde{G}^- = \sum_{k=1}^{l} \tilde{G}_k^-.$$

In the practical decision-making process, we usually think that the less the group inconsistency index, the better the decision-making result. And the group inconsistency index $\tilde{G}^-$ should not be larger than the group consistency index $\tilde{G}^*$. Therefore, the weight vector $\boldsymbol{w}$ of the attributes and the hesitant fuzzy positive ideal solution X^* should make the smallest while ensuring that $\tilde{G}^*$ is larger than $\tilde{G}^-$. Based on the above analysis, an optimization model can be established to determine the weight vector $\boldsymbol{w}$ of attributes and the hesitant fuzzy positive ideal solution X^*:

$$\min\{\tilde{G}^-\}$$
$$s.t.\begin{cases} \tilde{G}^* - \tilde{G}^- \geqslant \tilde{\varepsilon}, \\ 0 \leqslant (\gamma_i^{n_0})^* \leqslant (\gamma_i^{n_0-1})^* \leqslant \ldots \leqslant (\gamma_i^1)^* \leqslant 1,\ i = 1,2,\ldots,n, \\ 0 \leqslant w_j \leqslant 1, \sum_{j=1}^{n} w_j = 1,\ j = 1,2,\ldots,n, \end{cases}$$

where $\tilde{\varepsilon}$ is positive interval-value, which depends on the needs of the practical decision-making process, and $n_0 = |X^*|$ is the number of membership degrees in X^*.

After a series of transformations (Zhang and Xu, 2014a), the above model can be transformed into a single-objective linear programming model based on a parameter θ, where $\theta \in [0,1]$ is a given parameter which depends on the DMs.

The model can be solved by several mathematical programming models. Thus, the weight vector $\boldsymbol{w}$ of attributes and the hesitant fuzzy positive ideal solution X^* can be obtained. Then we can determine the orders of all alternatives under every expert by computing the distance between each alternative and X^*. Finally, the best alternative can be obtained with the Borda selection function or the Copeland selection function (Hwang and Yoon, 1981).

In addition, Zhang and Xu (2014a) also proposed another LINMAP method based on the hesitant fuzzy programming model. It can solve the MAGDM problems with the following conditions: The evaluation values and preference information between the alternatives given by the DMs are HFEs; The weights of attributes and the ideal solution are completely unknown. The ideas of the two methods are similar, so we will not go into details here.

5 Summary and outlook

5.1 The summary and supplement

Due to its own characteristics, HFSs have a very wide application in the process of solving GDM problems. This paper divides these applications into three parts: theory, support, and methods.

5.1.1 Theory

This part mainly includes the hesitant fuzzy aggregation operators and HFPRs, specifically the concepts, properties and operations of HFSs and HFEs, some different kinds of hesitant fuzzy aggregation operators, the definitions of HFPR and its extended forms HMPR and IV-HFPR. In addition to the content mentioned in the text, Zhu et al. (2012, 2013) proposed the hesitant fuzzy Bonferroni average (HFABM) and hesitant fuzzy Bonferroni geometric (HFGBM) operators. Subsequently, some hesitant fuzzy aggregation operators based on quasi-arithmetic averaging and derivation ideas were proposed (Xia et al., 2013; Liao and Xu, 2014a). In order to solve the aggregation problems of HFEs in the multiplicative form, Xia and Xu (2011a) introduced the hesitant multiplicative aggregation operators. Zhao et al. (2015) proposed the hesitant fuzzy prioritized "or" operator to solve the MADM problems that do not allow compromises between attributes. There are also some other aggregation operators (Wei, 2012; Zhang, 2013; Zhang et al., 2014; Zhao et al., 2014; Meng et al., 2015; Tan et al., 2015; Qin et al., 2016) that can be used under certain conditions, we will not repeat them here. The above operators are more able to adapt to complex situations than those mentioned in the text. They can provide more powerful guarantees for us to solve the hesitant fuzzy MAGDM problems.

5.1.2 Support

In the GDM problems, we always need to consider the opinions of all experts. Thus, when the gap in the opinions of different experts is obvious, the consensus becomes an important support for the GDM process. In the text, we firstly introduce a complete decision support model based on the consensus of HFPRs, which includes the establishment of consistency and consensus index, as well as feedback mechanisms for unacceptable HFPRs. Next, some supplementary studies were introduced, especially the consensus process under hesitant multiplicative environment. At last, as a typical application of the consensus process under hesitant multiplicative environment, hesitant fuzzy AHP was introduced. It is a great tool to solve hesitant fuzzy GDM problems. Besides these, Zhang et al. (2018c) derived the priority weight vector from an incomplete HFPR with the logarithmic least squares method and defined a novel consistency index of an incomplete HFPR. Xu et al. (2017a) proposed a dynamical weight adjustment method based on the consensus reaching process of HFPRs in the hesitant fuzzy GDM problems. In a word, there is not much research on the consensus process for hesitant fuzzy GDM. This field has great potential for development.

5.1.3 Methods

In Section 4, we introduced seven methods and some extensions of them based on hesitant fuzzy information for solving hesitant fuzzy GDM problems. These methods have their own characteristics and can solve the hesitant fuzzy GDM problems under different conditions. Besides, Yu et al. (2013) explored an aggregation method of prioritized HFEs and applied them to hesitant fuzzy GDM problems. Jin et al. (2016) established a programming model to determine the optimal weight of an attribute with some information measures of IVHPEs given by them, and proposed a hesitant fuzzy MAGDM method benefitted from the model. He et al. (2016a) studied HMPRs and proposed three methods for the priorities of HMPRs with the error-analysis technique. And then, they developed an approach to GDM with HMPRs by using those methods. He et al. (2016b) proposed two degrees to compare different IVHFSs. They also introduced two aggregation ways of IVHFEs with Bonferroni means, and applied those degrees and aggregation ways to the GDM problems under interval-valued hesitant fuzzy environment. Zhang et al. (2015b) proposed a method to solve the MAGDM problems with incomplete HFPRs based on the additive consistency of HFPRs. Wan et al. (2017) developed a mathematical programming method for solving the hybrid MAGDM problems with hesitant fuzzy truth degrees and the group consistency index of HFPRs. In addition to these, many scholars have proposed some HFSs-based GDM methods and applied them to all aspects of our daily life (Farhadina, 2014; Zhang and Wu, 2014a; Ashtiani and Azgomi, 2016; Li et al., 2015; Sevastjanov and Dymova, 2015; Farhadinia, 2016a; Perez-Fernandez et al., 2016; Xu et al., 2016; Zhang, 2016; De and Sana, 2017; Lan et al., 2017; Meng and An, 2017; Acar et al., 2018; Asan et al., 2018; Cheng, 2018; Osiro et al., 2018; Dincer et al., 2019). Moreover, several scholars have made minor improvements to the HFSs and the HFPRs to obtain some extensions of HFSs and HFPRs, and proposed some hesitant fuzzy GDM methods based on them, such as the multi-hesitant fuzzy sets (Peng et al., 2015), the hesitant-intuitionistic fuzzy sets and preference relations (Zhou et al., 2015), the hesitant Pythagorean fuzzy sets (Liang and Xu, 2017), the necessary and possible HFSs (Alcantud and Giarlotta, 2019) and the generalized HFSs (Peng et al., 2013; Qian et al., 2013).

The above content can explain the following two points: (1) HFSs and HFPRs do have a wide range of applications in group decision-making problems; (2) In recent years, more studies have gradually turned to some extensions of HFSs.

5.2 A brief introduction to probabilistic hesitant fuzzy sets and preference relations

As mentioned earlier, due to the need of practical decision-making problems, more and more scholars have begun to study some extended forms of HFSs. But the most important and practical one is the probabilistic hesitant fuzzy set (P-HFS). The emergence of P-HFS and P-HFE is to compensate for the defects of HFSs. In fact, due to its own characteristics, HFSs have the defect of losing information. Benefitted from applying the probability distribution, the P-HFSs can effectively avoid this problem. In this part, we introduce the relevant content of the P-HFS.

Zhu (2014) firstly proposed the concept of P-HFS and P-HFE. The core idea of P-HFS is to give each membership degree in HFEs a subordinate probability value that matches its own weight. After that, Zhang and Wu (2014b) studied some operations of P-HFEs and applied them to the hesitant fuzzy MADM problems. Farhadinia (2016b) introduced the similarity measure of P-HFEs and applied it to the field of medical diagnosis. Zhu and Xu (2018) compiled the related concepts and properties of P-HFS and introduced the definition of probabilistic hesitant fuzzy preference relation (P-HFPR). In the same paper, the consistency of P-HFPR has been preliminarily studied. Zhang et al. (2017) revised the definition of P-HFS and gave the calculations, ranking methods, standardization process and some aggregation operators of P-HFSs. Subsequently, the probabilistic interval-valued hesitant fuzzy preference relation (P-IVHFPR) (Zhang et al., 2018a) was proposed. At this point, the basic research for P-HFSs has basically been completed. In recent years, it also has some applications in the GDM problems.

Xu and Zhou (2017) established the consensus in the GDM process under probabilistic hesitant fuzzy environment. They also proposed the expected consistency of the P-HFPR, and used it to calculate the probabilities in the consistent P-HFPRs (Zhou and Xu, 2018). Wu et al. (2018) provided a new local feedback mechanism in a consensus process of P-HFPRs. Sometimes, the probabilities in P-HFEs or P-HFPRs may not easy to obtain, based on which, the uncertain P-HFE and uncertain P-HFPR were proposed. After that, the group consistency index and a GDM method under uncertain probabilistic hesitant fuzzy environment were provided (Zhou and Xu, 2017b). Moreover, Tian et al. (2018) established a prospect consensus process based on P-HFPRs and applied it to the field of venture capital. The above methods provide the important supports for the GDM processes under probabilistic hesitant fuzzy environment. In addition, Ding et al. (2017) proposed an interactive decision-making method for solving the GDM problems based on probabilistic hesitant fuzzy information. Zhang et al. (2018) proposed a method to deal with the GDM problems based on P-IVHFPRs. Jiang et al. (2017) introduced the probabilistic fuzzy regression approach. Based on which, they proposed a preference model under probabilistic fuzzy environment. Zhou and Xu (2017a) gave a tail decision-making method based on expected hesitant VaR under probabilistic hesitant fuzzy environment.

5.3 Prospects for future research directions

Based on the status of group decision-making under hesitant fuzzy environment, we believe that the future research directions about it are mainly divided into the following three aspects:

(1) Simplify the operations of HFEs: At this stage, the operations of HFEs are very complicated. Taking the simplest operation “addition” as an example, given two HFEs, each of them has five membership degrees. Then, there may be up to twenty-five membership degrees in the “sum” of them. Moreover, if another HFE with three membership degrees is added, the total “sum” may have up to seventy-five membership degrees. However, in the GDM problems, we often need to integrate the evaluation values given by multiple experts on multiple attributes. At this time, the complex operations become a huge obstacle. Therefore, the simplification of operations of HFEs is an urgent problem to be solved on the application of HFEs to GDM.

(2) The P-HFS and its application to the GDM: As an extension of HFS, the P-HFS has its unique advantages, especially in the GDM problems. For example, suppose that 20 DMs give their evaluation values to an alternative with respect to an attribute, where 5 of them give 0.5, 12 of them give 0.6, 3 of them give 0.7. Then, if we use an HFE to express those values, $h=\{0.5,0.6,0.7\}$ can be obtained. Obviously, it simply cannot reflect the weights behind the membership degrees. On the contrary, the P-HFE $h(p)=\{0.5(0.25),0.6(0.6),0.7(0.15)\}$ can more fully express the opinions of the DMs. Currently, research on P-HFE is still in its infancy. Therefore, GDM under the probabilistic hesitant fuzzy environment is also a potential research direction in the future.

(3) Combine multiple types of information: When we deal with the GDM problems with HFSs or HFPRs, due to certain conditions, we always hope that experts can give their evaluation values in the form of real numbers or HFEs. But in some cases, some experts can only give linguistic information or other forms of information. Thus, if the experts are forced to give their decision information as a certain form, it is likely to affect the accuracy and reliability of the decision-making results. Under these circumstances, we need some methods to aggregate multiple different types of information. Zhu et al. (2015) proposed a generalized analytic network process. In this method, the complex comparison matrices (CCMs) are used to collect the DMs’ preferences in multiple different forms, such as FPRs, IVFPRs, HFPRs and stochastic (stochastic fuzzy) preference relations. This laid a good foundation for the corresponding research. In a word, in future research, the combination of multiple types of data should be another focus of attention.

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